Output Regulation of Infinite-Dimensional Time-Delay Systems

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***Abstract*— We study output tracking and disturbance rejection for linear infinite-dimensional time-delay systems using dynamic error feedback controllers with state delays. The class of systems covers many partial differential equations with state, input, and output delays. As our main result we characterize the solvability of the control problem in terms of the solvability of the associated regulator equations.**

I. Introduction

We study the output regulation problem for an infinite-dimensional time-delay system of the form

$$\dot{x}(t) = \sum_{j=0}^{r} A_j x(t-\tau_j) + \sum_{j=0}^{r} B_j u(t-\tau_j) + B_d w(t), \quad \text{(1a)}$$

$$y(t) = \sum_{j=0}^{r} C_j x(t-\tau_j) + \sum_{j=0}^{r} D_j u(t-\tau_j) \quad \text{(1b)}$$

with $x(0) = x_0 \in X$ and $x(\cdot) = x_h(\cdot) \in L^2(-\tau_r, 0; X)$, where $0 = \tau_0 < \tau_1 < \cdots < \tau_r$. The main goal in the control problem is to choose a control law in such a way that the output $y(t)$ of the plant converges asymptotically to a reference signal $y_{ref}(\cdot)$ despite the external disturbance signal $w(\cdot)$. The signals $y_{ref}(\cdot)$ and $w(\cdot)$ are assumed to be generated by an *exosystem* of the form

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \quad \text{(2a)}$$

$$w(t) = Ev(t) \quad \text{(2b)}$$

$$y_{ref}(t) = -Fv(t). \quad \text{(2c)}$$

In the literature the output regulation problem for finite-dimensional time-delay systems has been considered most notably in the references [1], [2] in the case of static state feedback control, and in [3] with both state and error feedback control laws. In particular, in [3] the solvability of the output regulation problem was characterized in terms of solvability of the so-called *regulator equations* [4]. More recently, the output regulation problem for finite-dimensional delay systems was solved using an internal model based controller in [5] and using state predictors in [6].

In this paper we employ the operator theoretic methods used in [7], [8], [9] to study infinite-dimensional time-delay systems, such as linear partial differential equations with delays. As our main result we show that the output regulation problem is solvable with an error feedback controller with delays if and only if the infinite-dimensional regulator equations

$$\Sigma S = \sum_{j=0}^{r} A_{ej} \Sigma T_S(-\tau_j) + B_e \quad \text{(3a)}$$

$$0 = \sum_{j=0}^{r} C_{ej} \Sigma T_S(-\tau_j) + D_e \quad \text{(3b)}$$

have a solution $\Sigma \in \mathcal{L}(W, X_e)$. Here $((A_{ej}), B_e, (C_{ej}), D_e)$ are the parameters of the closed-loop system consisting of the plant and the controller, and $T_S(t)$ is the strongly continuous group generated by the system operator S of the exosystem. Our results generalize the characterization of the solvability of the output regulation presented in [3] in the case of finite-dimensional time-delay systems. Our results also allow the exosystem (2) to be an infinite-dimensional system. This generalization facilitates the study of nonsmooth periodic and almost periodic reference and disturbance signals [10].

The characterization of the solvability of the output regulation problem in terms of the solvability of (3) is based on the property that the solution of (3a) describes the steady state behaviour of the closed-loop system. If the system (1), the exosystem and the controller are finite-dimensional, the equation (3a) has a solution whenever the closed-loop system is stable and the eigenvalues of S have nonnegative real parts [3], [11]. For infinite-dimensional systems — and especially for infinite-dimensional exosystems — the solvability of the operator equation (3a) becomes an interesting mathematical problem. In this paper we present various sufficient conditions for the existence and uniqueness of solutions of (3a). We in particular show that (3a) has a unique solution whenever the closed-loop system is exponentially stable and the system operator S of the exosystem is a bounded operator.

The paper is organized as follows. In Section II we introduce notation and present the standing assumptions on the system, the exosystem, and the controller. In Section III we formulate the output regulation problem and present our main results concerning its solvability. Section IV is devoted to the question of solvability of (3a). Section V contains concluding remarks.

II. Mathematical Preliminaries and Standing Assumptions

A. Notation

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$ the domain and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$,

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then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space and the dual pairing of a Banach space are denoted by $\langle \cdot, \cdot \rangle$. For an ordered set of operators $(T_0, \ldots, T_r)$ we denote

$$T(\lambda) = \sum_{j=0}^{r} e^{-\lambda \tau_j} T_j.$$

B. The Plant

We assume the plant (1) is a time-delay system on a Banach space X, $A_0 : \mathcal{D}(A_0) \subset X \to X$ generates a strongly continuous semigroup $T_0(t)$ on X, and $A_j \in \mathcal{L}(X)$ for $j \in \{1, \ldots, r\}$. The input and output operators are assumed to be bounded in such a way that $(B_j) \subset \mathcal{L}(U, X)$, $B_d \in \mathcal{L}(U_d, X)$, $(C_j) \subset \mathcal{L}(X, Y)$, and $(D_j) \subset \mathcal{L}(U, Y)$ where U, U_d, and Y are Banach spaces. Under these assumptions the plant (1) has a well-defined mild solution [12, Ch. II], [13].

The transfer function $P(\lambda)$ (from $\hat{u}$ to $\hat{y}$) of the plant (1) is given by

$$P(\lambda) = C(\lambda) R(\lambda, A(\lambda)) B(\lambda) + D(\lambda),$$

for all $\lambda \in \mathbb{C}$ such that $0 \in \rho(\lambda - A(\lambda))$.

C. The Exosystem

We assume the exosystem (2) on a Hilbert space W is such that S generates a strongly continuous group $T_S(t)$ on W, $E \in \mathcal{L}(W, U_d)$, and $F \in \mathcal{L}(W, Y)$. We assume the exosystem satisfies a "nondecay condition" which requires that for any $Q \in \mathcal{L}(W, \mathbb{C})$

$$\lim_{t \to \infty} Q T_S(t) v_0 = 0 \quad \forall v_0 \in W \qquad \Leftrightarrow \qquad Q = 0. \tag{4}$$

In particular, any finite-dimensional exosystem with $\sigma(S) \subset \overline{\mathbb{C}_+}$ has the property (4).

D. The Controller

The controller we consider is of the form

$$\begin{aligned} \dot{z}(t) &= \sum_{j=0}^{r} \mathcal{G}_1^j z(t - \tau_j) + \mathcal{G}_2 e(t) \\ u(t) &= K z(t), \end{aligned}$$

with $z(0) = z_0 \in Z$ and $z(\cdot) = z_h \in L^2(-\tau_r, 0; Z)$ on a Banach space Z. Here $e(t) = y(t) - y_{\mathit{ref}}(t)$ is the *regulation error*. We assume $\mathcal{G}_1^0$ generates a strongly continuous semigroup on Z, $\mathcal{G}_1^j \in \mathcal{L}(Z)$ for $j \in \{1, \ldots, r\}$, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, and $K \in \mathcal{L}(Z, U)$.

Remark 2.1: We assume that the delays $\{\tau_j\}_j$ are the same for the state, the inputs, and the outputs of the system, and for the controller state. This does not result in any loss of generality since any of the operators A_j, B_j, C_j, D_j, and $\mathcal{G}_1^j$ for $j \in \{1, \ldots, r\}$ may be chosen to be zero operators.

E. The Closed-Loop System

The closed-loop system with state $x_e(t) = (x(t), z(t))^T$ on $X_e = X \times Z$ is a system with only state and output delays,

$$\dot{x}_e(t) = \sum_{j=0}^{r} A_{ej} x(t - \tau_j) + B_e v(t), \tag{5a}$$

$$e(t) = \sum_{j=0}^{r} C_{ej} x_e(t - \tau_j) + D_e v(t) \tag{5b}$$

with $x_e(0) = x_{e0} = (x_0, z_0)^T \in X_e$, $x_e(\cdot) = x_{eh} = (x_h, z_h)^T \in L^2(-\tau_r, 0; X_e)$, and for $j \in \{0, \ldots, r\}$

$$A_{ej} = \begin{bmatrix} A_j & B_j K \\ \mathcal{G}_2 C_j & \mathcal{G}_1^j + \mathcal{G}_2 D_j K \end{bmatrix}, \quad C_{ej} = \begin{bmatrix} C_j & D_j K \end{bmatrix}$$

$$B_e = \begin{bmatrix} B_d E \\ \mathcal{G}_2 F \end{bmatrix}, \qquad D_e = F.$$

The operator A_{e0} generates a strongly continuous semigroup $T_{e0}(t)$ on X_e, and the rest of the operators are bounded. Furthermore, we have $C_e(\lambda) = \begin{bmatrix} C(\lambda) & D(\lambda) K \end{bmatrix}$ and

$$A_e(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) K \\ \mathcal{G}_2 C(\lambda) & \mathcal{G}_1(\lambda) + \mathcal{G}_2 D(\lambda) K \end{bmatrix}.$$

III. The Output Regulation Problem

In this section we formulate the output regulation problem and characterize its solvability in terms of the regulator equations. In the study of output regulation with infinite-dimensional exosystems, exponential closed-loop stability may sometimes be unachievable. Because of this, we formulate the output regulation problem assuming *strong stability* of the closed-loop, meaning that if $v(t) \equiv 0$, then for all $x_{e0} \in X_e$, $x_{eh} \in L^2(-\tau_r, 0; X_e)$ the state $x_e(t)$ of the closed-loop system (5) satisfies $\|x_e(t)\| \to 0$ as $t \to \infty$. If the closed-loop system is exponentially stable, i.e., if there exist $M, \alpha > 0$ such that for $v(t) \equiv 0$ we have

$$\|x_e(t)\| \le M e^{-\alpha t} \left(\|x_{e0}\| + \|x_{eh}\|_{L^2} \right),$$

the proofs of the results show that also the convergence of the regulation error happens at an exponential rate. In particular, in this situation there exists $M' > 0$ such that

$$\|e(t)\| \le M' e^{-\alpha t} \left(\|x_{e0}\| + \|x_{eh}\|_{L^2} + \|v_0\| \right).$$

The Output Regulation Problem: Choose the parameters of the controller in such a way that the following are satisfied:

(a) The closed-loop system is strongly stable.

(b) For all $x_{e0} \in X_e$, $x_{eh} \in L^2(-\tau_r, 0; X_e)$, and $v_0 \in W$ the regulation error decays to zero asymptotically, i.e., $\|e(t)\| \to 0$ as $t \to \infty$.

Theorem 3.1 below characterizes the solvability of the output regulation problem in terms of the solvability of the regulator equations

$$\Sigma S = \sum_{j=0}^{r} A_{ej}\Sigma T_S(-\tau_j) + B_e \tag{6a}$$

$$0 = \sum_{j=0}^{r} C_{ej}\Sigma T_S(-\tau_j) + D_e. \tag{6b}$$

The theorem generalizes results in [3], [2], [11] for infinite-dimensional systems with delays. Theorem 3.1 assumes the solvability of the operator equation (6a). Later in Section IV we present conditions for this assumption to be satisfied. In particular, Theorem 3.1 is applicable whenever the closed-loop system is exponentially stable and the exosystem (2) is finite-dimensional.

Theorem 3.1: Assume that the closed-loop system is strongly stable, (6a) has a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$, and the exosystem satisfies the nondecay condition (4). Then the following are equivalent.

(a) The controller solves the output regulation problem.

(b) The equations (6) have a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$.

Remark 3.2: Part (b) implies part (a) even if the exosystem does not satisfy the nondecay condition (4).

The proof of Theorem 3.1 is based on the following lemma.

Lemma 3.3: Assume the closed-loop system is strongly stable, and assume (6a) has a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$. Then for all $x_{e0} \in X_e$, $x_{eh} \in L^2(-\tau_r, 0; X_e)$ and $v_0 \in W$ the mild state of the closed-loop system and the regulation error satisfy

$$\|x_e(t) - \Sigma v(t)\| \to 0 \qquad \text{as} \quad t \to \infty,$$

$$\|e(t) - (\sum_{j=0}^{r} C_{ej}\Sigma T_S(-\tau_j) + D_e)v(t)\| \to 0 \quad \text{as} \quad t \to \infty.$$

If the closed-loop system is exponentially stable, then the above convergences happen at uniform exponential rates with respect to $\|x_{e0}\|$, $\|x_{eh}\|_{L^2}$, and $\|v_0\|$.

Proof: Denote $\Delta(t) = x_e(t) - \Sigma v(t)$ for all $t \geq -\tau_r$. Then

$$\Delta(t) = T_{e0}(t)x_{e0} + \sum_{j=1}^{r} \int_0^t T_{e0}(t-s)A_{e1}x_e(s-\tau_j)ds$$
$$+ \int_0^t T_{e0}(t-s)B_eT_S(s)v_0 ds - \Sigma T_S(t)v_0.$$

Similarly as in [8], [9] we can show that since Σ is the solution of (6a), for every $v_0 \in \mathcal{D}(S)$ we have

$$\begin{aligned}
&\int_0^t T_{e0}(t-s)B_eT_S(s)v_0 ds \\
&\quad = \int_0^t T_{e0}(t-s)(\Sigma S - A_{e0}\Sigma)T_S(s)v_0 ds \\
&\qquad - \sum_{j=1}^{r} \int_0^t T_{e0}(t-s)A_{ej}\Sigma T_S(s-\tau_j)v_0 ds \\
&\quad = \Sigma T_S(t)v_0 - T_{e0}(t)\Sigma v_0 \\
&\qquad - \sum_{j=1}^{r} \int_0^t T_{e0}(t-s)A_{ej}\Sigma T_S(s-\tau_j)v_0 ds.
\end{aligned}$$

Since the operators on both sides of the equation are bounded and since $\mathcal{D}(S)$ is dense in W, the formula also holds for all $v_0 \in W$. Substituting the above expression into the formula for $\Delta(t)$ we obtain

$$\Delta(t) = T_{e0}(t)\Delta(0) + \sum_{j=1}^{r} \int_0^t T_{e0}(t-s)A_{ej}\Delta(s-\tau_j)ds.$$

Thus $\Delta(\cdot)$ is the mild solution of the delay equation

$$\dot{\Delta}(t) = \sum_{j=0}^{r} A_{ej}\Delta(t-\tau_j)$$

with $\Delta(0) \in X_e$ and $\Delta(\cdot) \in L^2(-\tau_r, 0; X_e)$. Since the closed-loop was assumed to be stable, we have $\|x_e(t) - \Sigma v(t)\| = \|\Delta(t)\| \to 0$ as $t \to \infty$ for all x_{e0}, x_{eh} and v_0. A direct computation shows that the corresponding regulation error satisfies

$$e(t) - \Big(\sum_{j=0}^{r} C_{ej}\Sigma T_S(-\tau_j) + D_e\Big)v(t) = \sum_{j=0}^{r} C_{ej}\Delta(t-\tau_j)$$

which immediately implies the second claim. ■

Proof of Theorem 3.1: If the equations (6) have a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$, then $\|e(t)\| \to 0$ as $t \to \infty$ follows immediately from Lemma 3.3 and (6b).

On the other hand, assume the controller solves the output regulation problem and the exosystem satisfies the nondecay condition (4). Let $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$ be a solution of (6a) and let $Q_0 \in \mathcal{L}(Y, \mathbb{C})$, x_{e0}, and x_{eh} be arbitrary. Lemma 3.3 implies that for all $v_0 \in W$

$$\begin{aligned}
&\|Q_0(\sum_{j=0}^{r} C_{ej}\Sigma T_S(-\tau_j) + D_e)v(t)\| \\
&\quad \leq \|Q_0\|\|e(t)\| \\
&\qquad + \|Q_0\|\|e(t) - \Big(\sum_{j=0}^{r} C_{ej}\Sigma T_S(-\tau_j) + D_e\Big)v(t)\| \to 0
\end{aligned}$$

as $t \to \infty$. The nondecay condition (4) implies

$$Q_0\Big(\sum_{j=0}^{r} C_{ej}\Sigma T_S(-\tau_j) + D_e\Big) = 0,$$

and since $Q_0 \in \mathcal{L}(Y, \mathbb{C})$ was arbitrary, we have that (6b) holds. ■

IV. Solvability of the Equation (6a)

In this section we present sufficient conditions for the solvability of the equation

$$\Sigma S = \sum_{j=0}^{r} A_{ej} \Sigma T_S(-\tau_j) + B_e \tag{7}$$

for different types of stability of the closed-loop system and for different types of exosystems. Our first result shows that the solution of (7) is unique whenever the closed-loop system is weakly stable, i.e., when for $v(t) \equiv 0$ the state $x_e(t)$ of the closed-loop system satisfies $|Q_0 x_e(t)| \to 0$ as $t \to \infty$ for all $x_{e0} \in X_e$, $x_{eh} \in L^2(-\tau_r, 0; X_e)$, and $Q_0 \in \mathcal{L}(X_e, \mathbb{C})$.

Lemma 4.1: It the closed-loop system is weakly stable and the exosystem satisfies the nondecay condition (4), then (7) can have at most one solution.

Proof: Let $\Sigma_1, \Sigma_2 \in \mathcal{L}(W, X_e)$ be two solutions of (7). The difference $\Delta = \Sigma_1 - \Sigma_2$ satisfies

$$\Delta S = \sum_{j=0}^{r} A_{ej} \Delta T_S(-\tau_j).$$

Let $v_0 \in W$ and $Q_0 \in \mathcal{L}(X_e, \mathbb{C})$ be arbitrary. A direct computation shows

$$\begin{aligned} &\Delta T_S(t) v_0 - T_{e0}(t) \Delta v_0 \\ &\quad - \sum_{j=1}^{r} \int_0^t T_{e0}(t-s) A_{ej} \Delta T_S(s - \tau_j) v_0 ds \\ &= \int_0^t T_{e0}(t-s)(\Delta S - \sum_{j=0}^{r} A_{ej} \Delta T_S(-\tau_j)) T_S(s) v_0 ds = 0, \end{aligned}$$

and thus $\Delta T_S(\cdot) v_0$ is the mild solution of a delay differential equation

$$\dot{x}_e(t) = \sum_{j=0}^{r} A_{ej} x_e(t - \tau_j).$$

Since the closed-loop system was assumed to be weakly stable, we have $|Q_0 \Delta T_S(t) v_0| \to 0$ as $t \to \infty$. Since this is true for all $v_0 \in W$, the nondecay condition (4) implies $Q_0 \Delta = 0$. Finally, since $Q_0 \in \mathcal{L}(X_e, \mathbb{C})$ was arbitrary, we must have $\Delta = 0$, and thus $\Sigma_1 = \Sigma_2$. ■

Theorem 4.2: Assume that $S \in \mathcal{L}(W)$ and assume the closed-loop is such that $0 \in \rho(\lambda - A_e(\lambda))$ for all $\lambda \in \sigma(S)$. Then (7) has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_{e0})$.

Proof: Let γ be a piecewise smooth closed positively oriented curve (possibly consisting of multiple parts) such that $\sigma(S)$ lies inside γ and the points $\lambda \in \mathbb{C}$ for which $0 \in \sigma(\lambda - A_e(\lambda))$ lie outside γ. We will show that the operator $\Sigma \in \mathcal{L}(W, X_e)$ with $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_{e0})$ defined by

$$\Sigma v = \int_\gamma R(\lambda, A_e(\lambda)) B_e R(\lambda, S) v d\lambda, \qquad \forall v \in W$$

is a solution of (7). Let $v \in W$ and let γ' be a piecewise smooth closed positively oriented curve such that γ lies inside γ' and the points $\lambda \in \mathbb{C}$ for which $0 \in \sigma(\lambda - A_e(\lambda))$ lie outside γ'. Denote $R_\lambda = R(\lambda, A_e(\lambda))$. For every $j \in \{1, \ldots, r\}$ the resolvent identity $(\lambda - \mu) R(\lambda, S) R(\mu, S) = R(\mu, S) - R(\lambda, S)$ implies

$$\begin{aligned} &\int_\gamma R(\lambda, A_e(\lambda)) B_e (e^{-\lambda\tau_j} - T_S(-\tau_j)) R(\lambda, S) v d\lambda \\ &\quad = \int_\gamma R_\lambda B_e (e^{-\lambda\tau_j} - \int_{\gamma'} e^{-\mu\tau_j} R(\mu, S) d\mu) R(\lambda, S) v d\lambda \\ &\quad = \int_\gamma R_\lambda B_e \left(e^{-\lambda\tau_j} - \int_{\gamma'} \frac{e^{-\mu\tau_j}}{\mu - \lambda} d\mu \right) R(\lambda, S) v d\lambda \\ &\qquad + \int_{\gamma'} \int_\gamma \frac{e^{-\mu\tau_j}}{\mu - \lambda} R_\lambda B_e R(\mu, S) v d\lambda d\mu = 0. \end{aligned}$$

Here the first term vanishes since $\int_{\gamma'} \frac{e^{-\mu\tau_j}}{\mu - \lambda} d\mu = e^{-\lambda\tau_j}$, and the inner integral of the second term is equal to zero since $\lambda \to \frac{e^{-\mu\tau_j}}{\mu - \lambda} R(\lambda, A_e(\lambda)) B_e R(\mu, S) v$ is analytic inside γ.

Since $R(\lambda, A_e(\lambda))$ is analytic inside γ and $R(\lambda, S) S = \lambda R(\lambda, S) - I$, the above equality implies

$$\begin{aligned} &\Sigma S v - \sum_{j=0}^{r} A_{ej} \Sigma T_S(-\tau_j) v = \int_\gamma R_\lambda B_e R(\lambda, S) S v d\lambda \\ &\qquad - \int_\gamma \sum_{j=0}^{r} A_{ej} R_\lambda B_e T_S(-\tau_j) R(\lambda, S) v d\lambda \\ &\quad = \int_\gamma (\lambda - \sum_{j=0}^{r} e^{-\lambda\tau_j} A_{ej}) R_\lambda B_e R(\lambda, S) v d\lambda = B_e v. \end{aligned}$$

Since $v \in W$ was arbitrary, Σ is a solution of (7).

To show that the solution is unique, assume Σ is a solution of (7). Let $\lambda \in \rho(S)$ such that $0 \in \rho(\lambda - A_e(\lambda))$. Then

$$\begin{aligned} &\Sigma(\lambda - S) = (\lambda - A_e(\lambda)) \Sigma - B_e \\ &\qquad + \sum_{j=1}^{r} A_{ej} \Sigma (e^{-i\omega_k\tau_j} - T_S(-\tau_j)) \\ \Leftrightarrow \quad &\Sigma R(\lambda, S) = R(\lambda, A_e(\lambda)) \Sigma + R(\lambda, A_e(\lambda)) B_e R(\lambda, S) \\ &\quad + \sum_{j=1}^{r} R(\lambda, A_e(\lambda)) A_{ej} \Sigma (T_S(-\tau_j) - e^{-i\omega_k\tau_j}) R(\lambda, S). \end{aligned}$$

Applying both sides of the above equation to $v \in W$ and integrating over γ yields

$$\begin{aligned} \Sigma v &= \int_\gamma R_\lambda B_e R(\lambda, S) v d\lambda \\ &\quad + \sum_{j=1}^{r} \int_\gamma R_\lambda A_{ej} \Sigma (T_S(-\tau_j) - e^{-i\omega_k\tau_j}) R(\lambda, S) v d\lambda \\ &= \int_\gamma R_\lambda B_e R(\lambda, S) v d\lambda, \end{aligned}$$

since the integrals in the sum vanish similarly as above. ■

Corollary 4.3: Assume the exosystem is finite-dimensional and the closed-loop is exponentially stable. Then (7) has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_{e0})$.

Corollary 4.4: Assume $S = \operatorname{diag}(i\omega_1, \ldots, i\omega_q)$, and $0 \in \rho(i\omega_k - A_e(i\omega_k))$ for all $k \in \{1, \ldots, q\}$. Then (7) has a unique solution given by

$$\Sigma v = \sum_{k=1}^{q} \langle v, \phi_k \rangle R(i\omega_k, A_e(i\omega_k)) B_e \phi_k, \qquad v \in W$$

where $\{\phi_k\}_{k=1}^q$ is the Euclidean basis of $W = \mathbb{C}^q$.

In the following theorem we assume the system operator S of the infinite exosystem on $W = \ell^2(\mathbb{C})$ is of the form

$$Sv = \sum_{k\in\mathbb{Z}} i\omega_k \langle v, \phi_k \rangle \phi_k, \tag{8a}$$

$$\mathcal{D}(S) = \left\{ v \in W \;\middle|\; (\omega_k \langle v, \phi_k \rangle)_k \in \ell^2(\mathbb{C}) \right\} \tag{8b}$$

where $(\phi_k)_{k\in\mathbb{Z}}$ is an orthonormal basis of W.

Theorem 4.5: Assume $W = \ell^2(\mathbb{C})$, S is as in (8), and the closed-loop is strongly stable in such a way that $0 \in \rho(i\omega_k - A(i\omega_k))$ for all $k \in \mathbb{Z}$. Then (7) has a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_{e0})$ if and only if

$$\sup_{\|x_e'\|\leq 1} \sum_{k\in\mathbb{Z}} \left| \langle R(i\omega_k, A_e(i\omega_k)) B_e \phi_k, x_e' \rangle \right|^2 < \infty. \tag{9}$$

If the solution exists, then it is unique and given by

$$\Sigma v = \sum_{k\in\mathbb{Z}} \langle v, \phi_k \rangle R(i\omega_k, A_e(i\omega_k)) B_e \phi_k, \qquad v \in W. \tag{10}$$

Proof: Similarly as in [8, Lem. 6] the operator $\Sigma : W \to X_e$ in (10) is bounded if and only if the condition (9) is satisfied. Let $\lambda \in \rho(A_{e0})$ and denote $R_\lambda = R(\lambda, A_{e0})$ and $R_k = R(i\omega_k, A_e(i\omega_k))$. For any $v \in \mathcal{D}(S)$ the identity $T_S(-\tau_j)^* \phi_k = e^{i\omega_k \tau_j} \phi_k$ implies that Σ in (9) satisfies

$$\begin{aligned} R(\lambda, A_{e0}) \Sigma (S - \lambda) v &= \sum_{k\in\mathbb{Z}} \langle v, \phi_k \rangle R_\lambda (i\omega_k - \lambda) R_k B_e \phi_k \\ &= \sum_{k\in\mathbb{Z}} \langle v, \phi_k \rangle \Big[-R_k + R_\lambda \sum_{j=0}^{r} e^{-i\omega_k \tau_j} A_{ej} R_k + R_\lambda \Big] B_e \phi_k \\ &= -\Sigma v + R_\lambda \sum_{j=1}^{r} A_{ej} \Sigma T_S(-\tau_j) v + R_\lambda B_e v. \end{aligned}$$

Since $v \in \mathcal{D}(S)$ was arbitrary, we have $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$ and Σ is a solution of (7). Finally, by Lemma 4.1 the solution is unique. ■

Corollary 4.6: Consider the infinite-dimensional exosystem (8), and assume $(E\phi_k)_{k\in\mathbb{Z}} \in \ell^2(X)$, $(F\phi_k)_{k\in\mathbb{Z}} \in \ell^2(Y)$, and $\sup_{k\in\mathbb{Z}} \|R(i\omega_k, A_e(i\omega_k))\| < \infty$. Then (7) has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_{e0})$ and given by the formula (10).

V. Conclusions

In this paper we have studied the output regulation problem for a infinite-dimensional time-delay systems. Our main interest has been in characterizing the solvability of the control problem using a dynamic error feedback controller with state delays. The most important topic for future research is the construction of controllers for output tracking and disturbance rejection.

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