



Author(s) Seppänen, Mari; Perälä, Tommi; Piché Robert

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Autonomous Satellite Orbit Prediction

Mari Seppänen, Tommi Perälä and Robert Piché
Tampere University of Technology, Finland

BIOGRAPHY

Mari Seppänen received the M.Sc. degree from Tampere University of Technology in March 2010. From then on, she has pursued the Ph.D. degree. Her current research interests include mathematical algorithms for satellite orbit prediction.

Tommi Perälä received his M.Sc. degree at Tampere University of Technology, Department of Mathematics in 2008 and is currently Ph.D. student at the same department. He has been working with personal positioning algorithms since 2005.

Robert Piché earned a Ph.D. in civil engineering from the University of Waterloo (Canada) in 1986. In 2004 he became full professor of mathematics at the Tampere University of Technology. His scientific interests are in applications of systems theory and numerical analysis to electromagnetics, hydraulic power systems, optics, structural mechanics, positioning/navigation, and finance.

ABSTRACT

A method to predict satellite orbits in a GPS device without a network connection is presented. The motivation for this work was to reduce time to first fix when assistance data is not available.

The orbit of a satellite is predicted by numerically integrating the differential equation that models its motion. The initial position and velocity values used in prediction correspond to those received from the broadcast when the device was last operated. These initial values are given in the Earth centered, Earth fixed reference frame and have to be transformed into an inertial reference frame prior to substitution into the equation of motion and subsequent integration. For this purpose, one needs to predict the movement of Earth's rotation axis with respect to both space (nutation and precession) and to the Earth's crust (polar motion). Using precise ephemeris as the initial condition, we found that this kind of model gave quite accurate prediction results.

However, the results were worse when initial conditions were computed from the less accurate broadcast ephemeris which, unfortunately, is the only ephemeris available to the navigation device without a network connection. In

addition, we were not able to find a model that would be able to predict Earth's polar motion with sufficient accuracy within the assumed lifetime of the device. Without the polar motion parameters, one cannot do the transformation from ECEF to an inertial reference frame. In this paper we will present a method to improve the accuracy of the initial velocity of the satellite computed from the broadcast and simultaneously solve the unknown polar motion parameters. Tests of our algorithm show that in 95% of the cases the error in satellite's predicted position stays under 21 meters for one day and under 94 meters for three days.

1 INTRODUCTION

In Global Positioning System (GPS) each satellite transmits its position in the form of ephemeris parameters and its clock time. The user's receiver measures the time delay it takes for the signal to travel from the satellite to the receiver, from which the apparent range to the satellite can be calculated. This range still includes an unknown bias originating from the time difference between the receiver's clock time and GPS time. When the biased range measurements and positions from four satellites are known simultaneously the unknown bias as well as the position of the user can be solved.

When a GPS navigation device is turned on, it typically takes about 30 seconds to get the first position estimate, even in an ideal environment. This delay occurs because the part of broadcast containing the ephemeris takes 12 seconds to transmit and the satellite sends it once every 30 seconds. When the receiver is without a straight view to the sky, for example blocked by trees or high buildings, the signal acquisition and demodulation slows down and it takes much longer to get the first position estimate. If the signal gets too weak, the demodulation is no longer even possible. As little time as this 30 seconds delay may seem, it is usually frustrating even from a typical user's point of view, to say nothing of emergency call cases or other special situations. Therefore some innovations are needed to provide the first position estimate in a significantly shorter time. That time is often discussed with abbreviation TTFF (time to first fix).

Because the main reason for the long TTFF is the time it takes to receive the satellites' ephemeris broadcast, alternative ways of obtaining satellite position information

can be used to reduce it. If the current ephemerides were in the device already before turning it on, the connection between satellite and the receiver would be needed only for the range measurements. This is fast, because the receiver is potentially capable of getting a new pseudorange measurement at the beginning of each subframe of the broadcast or every 6 seconds. Thus it would be possible to get the first position fix often under 5 seconds after turning on the device, if we only knew the positions of the satellites.

Besides reducing TTFB, satellite orbit prediction can also widen the capability of GPS devices. In some navigation cases, the user is indoors or in urban canyons where the signal is strong enough to be detected and for getting pseudorange measurements, but too weak or fragmental for reading the whole ephemeris. Then the predicted orbit can make it possible to both compute a position and reduce TTFB. Orbit prediction can also be used to find the visible satellites when the almanac is too old. Furthermore, if the velocities one gets from orbit prediction are accurate then the Doppler frequency shift can be estimated more accurately. These facts can speed up the acquisition in certain cases.

A widely used alternative to satellites' ephemeris broadcast is the use of assistance data servers that send data to the navigation device that enable the computation of satellites' position coordinates. However, there are problems associated with such assistance data: Connection to the assistance server may fail or the assistance data connection may be too expensive for the user. Furthermore, many navigation devices are designed to operate without any network connection. There is therefore interest in methods that can be implemented entirely in the navigation device, without network connection.

One such stand-alone technique is to compute the present satellite position using satellite broadcast data that was received when the device was in operation earlier. This technique has been implemented in commercial products and is outlined in the literature [1; 2], but these publications do not give a detailed description of the algorithms. The aim of this paper is to give a fully detailed presentation of an ephemeris prediction algorithm. Specifically, we present a detailed description of all the physical phenomena we have taken into account in our model as well as either introductions or references to the algorithms we have used.

We measure the performance of our model by calculating the prediction error or 3D-distance between predicted satellite position and precise ephemeris orbit. The navigation device's actual position accuracy depends apart from satellite orbit error on other matters like satellite geometry. The largest error source in this case might be the unknown time difference between GPS time and satellites'

clocks. In order to enable the evaluation of the GPS time, satellites send clock correction terms as a part of their broadcast. However when predicting orbits, we have to use old correction terms. In this paper we concentrate ourselves on satellite's position error and do not consider positioning error at all.

The remainder of this paper is organized as follows. In Section 2 we present the equation of motion used in this work to predict the orbit of a GPS satellite. In Section 3 we provide a detailed discussion on the different reference frames and the associated coordinate transformations involved in the calculations. In Section 4 we show how to use the equation of motion and the received ephemeris broadcasts to compute the initial velocity of the satellite and solve the unknown polar motion parameters simultaneously. Finally, in Sections 5 and 6 we present the results of our simulations and conclude.

2 FORCE MODEL

In this work the orbits of a GPS satellite is predicted by forming its equation of motion and solving it numerically. By Newton's second law $\mathbf{F} = m\ddot{\mathbf{r}}$, where \mathbf{F} is the vector sum of forces acting on the satellite, m is the mass of the satellite and $\ddot{\mathbf{r}}$ is the second time derivative of satellite's position. The position coordinate vector \mathbf{r} must be in an inertial reference frame for Newton's second law to hold true. Dividing through by m and denoting the acceleration by \mathbf{a} we get

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} = \mathbf{a}(t, \mathbf{r}) \quad (1)$$

Given the satellite's position and velocity at a moment t_0 , say $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$, we can compute the satellite's position at any other moment t by integrating equation (1) as follows:

$$\dot{\mathbf{r}}(t) = \mathbf{v}_0 + \int_{t_0}^t \left(\mathbf{v}_0 + \int_{t_0}^t \mathbf{a}(t, \mathbf{r}) dt \right) dt. \quad (2)$$

The largest forces acting on a GPS satellite are the Earth gravitation (taking into account the unsymmetrical mass distribution of the Earth), lunar gravitation, solar gravitation and solar radiation pressure. In our model, \mathbf{F} is the vector sum of these four forces and thus acceleration can be represented as

$$\mathbf{a}(t, \mathbf{r}) = \mathbf{a}_g + \mathbf{a}_{\text{moon}} + \mathbf{a}_{\text{sun}} + \mathbf{a}_{\text{srp}} \quad (3)$$

where functions \mathbf{a}_g , \mathbf{a}_{moon} , \mathbf{a}_{sun} and \mathbf{a}_{srp} correspond the accelerations due to Earth gravitation, lunar gravitation, solar gravitation and solar radiation pressure respectively.

We now consider a satellite as a point mass located at \mathbf{r} in an inertial Earth centered reference frame. Let $\rho(\mathbf{r}^*)$ be the mass density at a point \mathbf{r}^* inside the Earth and let

G be the gravitation constant. Then the part of satellite's acceleration due to Earth's gravitation is

$$\mathbf{a}_g(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}^*)(\mathbf{r} - \mathbf{r}^*)}{\|\mathbf{r} - \mathbf{r}^*\|^3} d\mathbf{r}^*.$$

Because $\nabla(1/\|\mathbf{r} - \mathbf{r}^*\|) = -(\mathbf{r} - \mathbf{r}^*)/\|\mathbf{r} - \mathbf{r}^*\|^3$, the equation can be reformulated as

$$\mathbf{a}_g(\mathbf{r}) = \nabla U(\mathbf{r}) = \nabla G \int \frac{\rho(\mathbf{r}^*)}{\|\mathbf{r} - \mathbf{r}^*\|} d\mathbf{r}^*,$$

where U is the gravity potential.

It is appropriate to write the potential U in Earth centered, Earth fixed reference frame. The advantage in this is that the Earth's density function ρ and therefore also U are time independent in a reference frame that is rotating with the Earth whereas in the equations above one should have actually written $\rho(\mathbf{r}^*, t)$. Denoting the satellite's ECEF position \mathbf{r}_e and the point inside the Earth \mathbf{r}_e^* the gravity potential can be written as

$$U(\mathbf{r}_e) = G \int \frac{\rho(\mathbf{r}_e^*)}{\|\mathbf{r}_e - \mathbf{r}_e^*\|} d\mathbf{r}_e^*. \quad (4)$$

If the \mathbf{r}_e was computed from corresponding inertial vector by multiplying by a transformation matrix \mathbf{R} i.e. $\mathbf{r}_e = \mathbf{R}\mathbf{r}$ then

$$\mathbf{a}_g(\mathbf{r}) = \mathbf{R}^{-1} \nabla U(\mathbf{r}_e). \quad (5)$$

However, before we can use this equation we first have to compute the gradient of the potential U .

In geodesy and related sciences it is customary to write the potential U as a spherical harmonics series. In this kind of representation, in place of the unobservable quantity ρ the Earth's mass distribution is represented by coefficients C_{nm} and S_{nm} , that can be determined experimentally. Furthermore in this representation U can be expressed as a series of terms that are functions of the longitude λ , latitude φ and radius $r = \|\mathbf{r}_e\|$ which are the spherical coordinates representation of the satellite's position \mathbf{r}_e . The spherical harmonics representation of the potential U is

$$U(\mathbf{r}_e) = \frac{GM_E}{r} \sum_{n=0}^{\infty} \sum_{m=0}^n \left[\left(\frac{R_E}{r} \right)^n P_{nm}(\sin \varphi) \left(C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda) \right) \right], \quad (6)$$

where P_{nm} is associated Legendre polynomial of degree n and order m , M_E is the mass and R_E is the radius of Earth. The derivation of this representation from the integral (4) can be found for example from the book [3, s. 56-57].

The first term of the sum (6) is the potential of Earth when approximating it as a spherically symmetric ball. The subsequent terms are added to correct this approximation.

Because they decrease very fast with increasing n and m , the potential can be approximated by taking into account only the first few terms. We have been using terms up to the degree and order 8.

The associated Legendre polynomials satisfy some recurrence relations that can be used to evaluate the terms in the sum (6). Furthermore also the partial derivatives of the potential U or the satellite accelerations corresponding to each potential term can be computed recursively. We have used the algorithm introduced by L. E. Cunningham [4] and later extended by Métris [5] for partial derivatives of higher degree.

After the Earth's gravitation the second biggest acceleration components in the satellite's equation of motion are caused by the gravitational forces of the Moon and the Sun. When computing these accelerations one must take into account that not only the satellite but also the Earth is drawn by the Moon. Therefore when dealing with Earth centered reference frame one has to compute the acceleration of the satellite in relation to the acceleration of the Earth. To compute this relative acceleration acting on the satellite because of the gravitational force of any celestial body, one can use the form

$$\mathbf{a}_{cb} = GM \left(\frac{\mathbf{r}_{cb} - \mathbf{r}}{\|\mathbf{r}_{cb} - \mathbf{r}\|^3} - \frac{\mathbf{r}_{cb}}{\|\mathbf{r}_{cb}\|^3} \right), \quad (7)$$

where M is the mass of the celestial body, \mathbf{r}_{cb} is its position in Earth centered inertial reference frame and \mathbf{r} is the position of the satellite in the same reference frame. Besides the Moon and the Sun this formula can be used also for computing the planetary accelerations, which we have ignored because of their small influence to the satellite's orbit.

The orbits of the Sun and the Moon have to be known in order to compute the gravitational acceleration with the formula (7). In our model we use simple models presented in [3, s. 70-73] to compute the lunar and the solar coordinates. The coordinates are accurate to about 0.1-1% [3].

The last acceleration component in equation (3) arises when a satellite reflects and absorbs photons emitted by the Sun. Taking into account only the part of radiation being parallel to the direction from satellite to Sun the formula

$$\mathbf{a}_{srp} = -\alpha \lambda P_0 (1 + \epsilon) \frac{AU^2}{r_{sun}^2} \frac{A}{m} \mathbf{e}_{sun} \quad (8)$$

gives an estimate for the acceleration. This kind of model for solar radiation pressure (SRP) is called also the Cannonball model [6]. In formula (8) the factor r_{sun} is the distance from satellite to Sun and \mathbf{e}_{sun} is a unit vector from satellite to Sun. The factor λ is a shadow function, whose value equals one when satellite is in sunlight, zero

when it is in umbra and something between when it is in penumbra. We have used a conical shadow model described in the book [3, s. 80-83]. The remaining factors are constants: AU is the astronomical unit, P_0 is the solar radiation pressure at the distance of 1AU from the Sun, ϵ is the reflectivity coefficient of the satellite, m is the mass of the satellite and A is the satellite's surface area. For these constants we have used values shown in the Table 1.

Table 1: Constants in the solar radiation pressure formula

P_0 [Nm ⁻²]	ϵ	AU [km]	A [m ²]	m [kg]
$4.56 \cdot 10^{-6}$	0.21	149 597 870.691	13.4	1075

However, it is hard to know the exact mass, surface area or reflectivity of the satellite. These numbers are also different for different satellite types or blocks IIR and IIA. For these reasons, the acceleration formula (8) is multiplied with an additional parameter α . When the value of this parameter is estimated based on real GPS orbit data, the magnitude of \mathbf{a}_{srp} gets fixed regardless of the values chosen for the other constants. We estimated α separately for each satellite using an extended Kalman filter, in which the measurement model is discrete and state model is continuous. This kind of filter is presented in [7, s. 278] and [8, s. 405]. Details of the state and measurement models for this case are presented in [9]. As measurement data we used precise ephemeris positions published by National Geospatial-Intelligence Agency (NGA) [10]. The estimation of the parameter alpha was done several times using different periods during the GPS weeks 1513 - 1568. The resulted values were varying a little bit as a function of time, especially during times when the satellite is in umbra. However we wanted a constant parameter for each satellite and therefore took the satellite-specific mean. Results of this estimation process are shown in Table 2.

Table 2: Solar radiation pressure parameters

PRN	1	2	3	4	5	6	7	8
α	1.43	1.47	1.33	1.34	1.44	1.34	1.44	1.33
	9	10	11	12	13	14	15	16
	1.33	1.32	1.47	1.44	1.48	1.48	1.45	1.48
	17	18	19	20	21	22	23	24
	1.43	1.49	1.46	1.48	1.45	1.47	1.50	1.36
	25	26	27	28	29	30	31	32
	1.34	1.33	1.33	1.46	1.42	1.33	1.45	1.35

3 REFERENCE FRAMES

The International Earth Rotation and Reference Systems Service (IERS) maintains two important reference systems: a Celestial Reference System (CRS) and a Terrestrial Reference System (TRS). CRS is an inertial reference system, whose coordinate axes maintain their orientation with respect to distant stars. However the origin of this

reference frame is in the center of the Earth and Earth is in an accelerated movement while orbiting around the sun. Therefore it is not precisely inertial, but a good approximation of an inertial reference frame. TRS is an Earth fixed reference frame. Its origin is the Earth's centre of mass and z-axis is the mean rotational axis of the Earth. This mean pole of rotation was defined, because the Earth's instantaneous rotation pole moves with respect to Earth's crust whereas in Earth fixed reference frame the axes must be pointing at a fixed point on the Earth's surface. The coordinate transformation between these two reference systems can be written as

$$\mathbf{r}_{\text{TRS}}(t) = \mathbf{W}(t)\mathbf{G}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{r}_{\text{CRS}}, \quad (9)$$

where the matrices \mathbf{W} , \mathbf{G} , \mathbf{N} and \mathbf{P} describe polar motion, Earth rotation, nutation and precession, respectively. The transformation matrices are time dependent and thus the vector \mathbf{r}_{CRS} , being constant in CRS, is time dependent after transformation to TRS. We use IAU76 theory when computing the precession matrix \mathbf{P} and nutation theory IAU80 for matrix \mathbf{N} . A detailed description of these models are presented in [3].

The third rotation matrix \mathbf{G} describes the rotation of the Earth. In order to compute it, we need Greenwich Mean Sidereal Time (GMST), which can be computed as follows. Starting from GPS time, we first compute corresponding Julian date as $JD_{\text{GPS}} = 2444244.5 + t/86400$ s, where t is the amount of seconds elapsed since the beginning of GPS time. After that we present the Julian date in Universal Time UT1 by subtracting the leap seconds τ which have been added to the Coordinated Universal Time (UTC) since the beginning of GPS time, 6.1.1980 and adding the difference $dUT1 = UT1 - UTC$. That is

$$JD_{\text{UT1}} = JD_{\text{GPS}} + \frac{dUT1 - \tau}{86400s}.$$

The amount of leap seconds is typically known by the GPS device, because the current amount of leap seconds is part of the broadcast message and new leap seconds are added quite seldom. However the time difference $dUT1$, which is one of the Earth orientation parameters (EOP), is not necessarily known when starting the prediction. This time difference is small ($|dUT1| < 0.9$ s), but it causes some error if we neglect it. When JD_{UT1} is divided into two parts, such that the first part is the JD at the beginning of the current day and the second part is the rest of JD_{UT1} , we get parameters $JD_{0^h\text{UT1}}$ and $UT1$. When computing these, one has to notice that the julian date number changes at noon, but 0 h universal time is at the midnight. Also the $UT1$ must be given in seconds. Taking these facts into account we can apportion as follows:

$$\begin{aligned} JD_{0^h\text{UT1}} &= \lfloor JD_{\text{UT1}} + 0.5 \rfloor - 0.5 \\ UT1 &= (JD_{\text{UT1}} - JD_{0^h\text{UT1}}) \cdot 86400 \text{ s}. \end{aligned}$$

Now we can compute the Greenwich Mean Sidereal Time in seconds with the formula

$$\text{GMST} = 24110.54841 + 1.002737909350795 \text{ UT1} + 8640184.812866 T_0 + 0.093104 T^2 - 6.2 \cdot 10^{-6} T^3, \quad (10)$$

where the time arguments T and T_0 are

$$T = \frac{\text{JD}_{\text{UT1}} - 2451545}{36525} \quad T_0 = \frac{\text{JD}_{0^h \text{UT1}} - 2451545}{36525},$$

i.e the number of Julian centuries of Universal Time elapsed since 2000 Jan. 1.5 UT1 at the current time and at the beginning of the day, respectively. Furthermore the equation of equinoxes

$$\text{GAST} = \text{GMST} + \Delta\psi \cos \varepsilon + 0.''002649 \sin \Omega + 0.''000013 \cos \Omega \quad (11)$$

is used to compute Greenwich Apparent Sidereal Time (GMST). In this equation the parameters $\Delta\psi$, ε and Ω come from the nutation theory and $''$ denotes that the number is presented in arcseconds. Finally, we can compute the transformation matrix

$$\mathbf{G}(t) = \mathbf{R}_z(\text{GAST}).$$

In this equation \mathbf{R}_z is a simple rotation around the z-axis i.e

$$\mathbf{R}_z(\gamma) = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

After multiplying the vector \mathbf{r}_{CRS} with precession, nutation and Earth rotation matrices, it is in a Terrestrial Intermediate Reference System (TIRS), whose z-axis points to the Celestial Ephemeris Pole (CEP). We assume that the orientation of Earth's rotation axis is the same as the orientation of this CEP pole. CEP is not fixed with respect to the surface of the Earth, but performs a periodic motion around its mean position, called polar motion. The motion is small, having a radius of under 10 m, but it is important to take this into account. The rotation matrix describing the polar motion is

$$\mathbf{W}(t) = \mathbf{R}_y(-x_p)\mathbf{R}_x(-y_p).$$

where x_p and y_p are the polar motion parameters and \mathbf{R}_x and \mathbf{R}_y are simple rotation matrices around the x- and y-axes. Together with dUT1 they are called also Earth Orientation Parameters (EOP). The daily values for these parameters can be found from the homepage of IERS [11].

IERS reports the observed values for EOP and quite accurate short term predictions. However the Earth orientation parameters are not long-term predictable, which causes some problems while trying to do the

prediction in a device without any network connection. We can not form a prediction model which would be valid for life of the device i.e. years. Neither are the EOP parameters part of the broadcast message yet, though this fact will be changed in the future as the new LIC signal comes into use [12]. However in chapter 4 we will show how to infer these parameters based on collected broadcast ephemeris data.

When transforming the position vector \mathbf{r}_{CRS} to the TRS one matrix at a time according to the equation (9), every intermediate step is also a reference frame. For example when multiplying with matrix \mathbf{P} we get a mean of date (mod) system and after multiplied also with nutation matrix \mathbf{N} we get a true of date (tod) system. The Figure 1 illustrates the connection between CRS and TRS, as well as the intermediate reference frames between them. Next we will present the reference frames we have used, and illustrate how the transformation matrices between them are compounded of those four matrices connecting CRS and TRS.

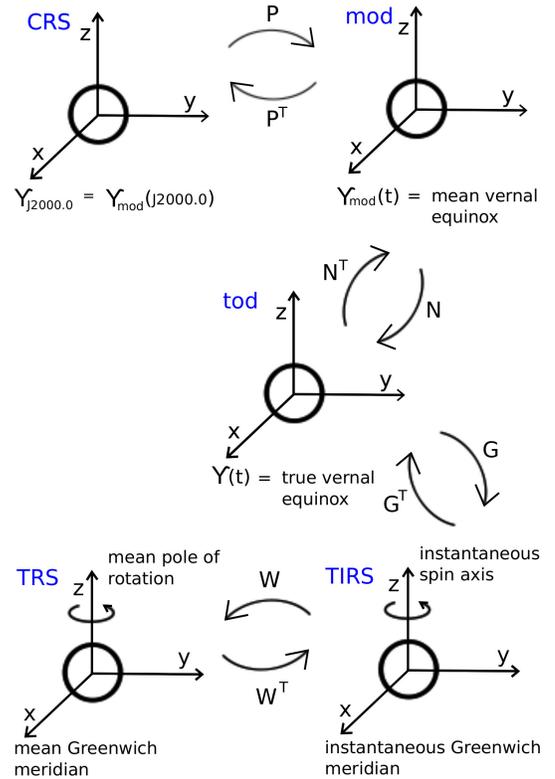


Figure 1: Connection between the IERS reference systems CRS and TRS. Transformation matrices and intermediate reference frames. The inverse of a transformation matrix is same as its transpose.

The broadcast position and velocity we get from broadcast ephemeris are in Earth fixed reference frame WGS84, but we assume those coordinates are equivalent to the corresponding TRS coordinates. Now for numerical integration we need an inertial reference frame and we choose to use the TIRS system at epoch t_0 , the initial

epoch for the equation of motion. Before starting to predict satellite's position according to the integral (2), we need to transform the position and velocity vectors to this inertial reference frame, denoted by subscript IN. For position vector, the transformation from TRS at an arbitrary time t to the TIRS system at epoch t_0 is

$$\begin{aligned}\mathbf{r}_{\text{IN}} &= \mathbf{r}_{\text{TIRS}(t_0)} \\ &= \mathbf{G}(t_0)\mathbf{N}(t_0)\mathbf{P}(t_0)\mathbf{P}^T(t)\mathbf{N}^T(t)\mathbf{G}^T(t)\mathbf{W}^T(t)\mathbf{r}_{\text{TRS}}.\end{aligned}\quad (13)$$

However, when we start the prediction we have $t = t_0$ and the transformation is simplified to

$$\mathbf{r}_{\text{IN}} = \mathbf{W}^T(t_0)\mathbf{r}_{\text{TRS}(t_0)}.\quad (14)$$

To make the transformation for velocity vector \vec{v}_{TRS} , we first derive the time derivative of the matrix \mathbf{G}^T . Denoting $\text{GAST} = \gamma(t)$ and using the chain rule we get

$$\frac{d\mathbf{G}^T(t)}{dt} = \frac{d\mathbf{R}_z^T(\gamma(t))}{d\gamma} = \frac{d\mathbf{R}_z^T(\gamma)}{d\gamma} \frac{d\gamma}{dt}\quad (15)$$

The derivative $d\mathbf{R}_z^T/d\gamma$ can be calculated using the equation (12)

$$\begin{aligned}\frac{d\mathbf{R}_z^T(\gamma)}{d\gamma} &= \begin{pmatrix} -\sin \gamma & -\cos \gamma & 0 \\ \cos \gamma & -\sin \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}\quad (16)$$

Multiplying a vector from the left side with the right most matrix is the same as cross product with $\mathbf{e}_z = [001]^T$. This is a reason for denoting this matrix ($\mathbf{e}_z \times$) in the following equation:

$$\frac{d\mathbf{R}_z^T(\gamma)}{d\gamma} = \mathbf{R}_z^T(\gamma)(\mathbf{e}_z \times).$$

Next step is to differentiate the $\text{GAST} (= \gamma)$ with respect to the time. Taking a look at the equations (10) and (11) one can notice that the time derivative of the term containing $UT1$ is the largest and the other terms are negligible. Therefore

$$\begin{aligned}\frac{d\gamma}{dt} &= \frac{d\text{GAST}}{dt} \approx 1.002737 \frac{2\pi}{86400\text{s}} \\ &= 7.2921158553 \cdot 10^{-5} \text{s}^{-1},\end{aligned}$$

which is the Earth's angular velocity and will be denoted as ω . Now according to equation (15)

$$\frac{d\mathbf{G}^T}{dt} = \mathbf{R}_z^T(\gamma)(\mathbf{e}_z \times) \cdot \omega = \mathbf{G}^T(\omega \times),$$

where $\omega = [0 \ 0 \ \omega]^T$.

Now we can calculate the transformation for a velocity vector by differentiating both sides of the equation

(13). Before doing that we denote the matrix product $\mathbf{G}(t_0)\mathbf{N}(t_0)\mathbf{P}(t_0)\mathbf{P}^T(t)\mathbf{N}^T(t)$ as \mathbf{C} and discover that the time derivative of this matrix product is close to zero. This is because the matrices at epoch t_0 are all constants and precession and nutation are much slower than Earth's rotation. Therefore we consider the matrix \mathbf{C} a constant when we now calculate the time derivative of the equation (13). We get

$$\begin{aligned}\frac{d\mathbf{r}_{\text{IN}}}{dt} &= \frac{d}{dt} (\mathbf{C}\mathbf{G}^T(t)\mathbf{W}^T(t)\mathbf{r}_{\text{TRS}}) \\ &= \frac{d}{dt} (\mathbf{C}\mathbf{G}^T(t)\mathbf{r}_{\text{TIRS}}) \\ &= \mathbf{C} \left(\mathbf{G}^T(t) \frac{d\mathbf{r}_{\text{TIRS}}}{dt} + \frac{d\mathbf{G}^T(t)}{dt} \mathbf{r}_{\text{TIRS}} \right) \\ &= \mathbf{C} (\mathbf{G}^T(t)\mathbf{v}_{\text{TIRS}} + \mathbf{G}^T(t)(\omega \times \mathbf{r}_{\text{TIRS}})).\end{aligned}$$

If the derivative of the polar motion matrix is assumed to be zero, then

$$\mathbf{v}_{\text{TIRS}} = \frac{d}{dt} (\mathbf{W}^T \mathbf{r}_{\text{TRS}}) = \mathbf{W}^T \mathbf{v}_{\text{TRS}}.$$

We get

$$\mathbf{v}_{\text{IN}} = \mathbf{C}\mathbf{G}^T(t) (\mathbf{W}^T(t)\mathbf{v}_{\text{TRS}} + \omega \times (\mathbf{W}^T(t)\mathbf{r}_{\text{TRS}})).$$

Next we denote the transformation matrix from reference frame A to B as \mathbf{R}_A^B . Now $\mathbf{R}_{\text{CRS}}^{\text{TIRS}} = \mathbf{G}\mathbf{N}\mathbf{P}$ and $\mathbf{R}_{\text{TIRS}}^{\text{CRS}} = (\mathbf{R}_{\text{CRS}}^{\text{TIRS}})^T = \mathbf{P}^T\mathbf{N}^T\mathbf{G}^T$. We end up with the formula

$$\begin{aligned}\mathbf{v}_{\text{IN}} &= \mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t_0)\mathbf{R}_{\text{TIRS}}^{\text{CRS}}(t) \left(\mathbf{W}^T(t)\mathbf{v}_{\text{TRS}} + \right. \\ &\quad \left. \omega \times (\mathbf{W}^T(t)\mathbf{r}_{\text{TRS}}) \right).\end{aligned}\quad (17)$$

for velocity transformation from TRS to the chosen inertial reference frame.

After predicted the satellite's position and velocity at time t in the future, we have to do the transformations the other way around. Solving \mathbf{r}_{TRS} and \mathbf{v}_{TRS} from equations (13) and (17) we get

$$\begin{aligned}\mathbf{r}_{\text{TRS}} &= \mathbf{W}(t)\mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t)\mathbf{R}_{\text{TIRS}}^{\text{CRS}}(t_0)\mathbf{r}_{\text{IN}} \\ \mathbf{v}_{\text{TRS}} &= \mathbf{W}(t) \left(\mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t)\mathbf{R}_{\text{TIRS}}^{\text{CRS}}(t_0)\mathbf{v}_{\text{IN}} - \right. \\ &\quad \left. \omega \times (\mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t)\mathbf{R}_{\text{TIRS}}^{\text{CRS}}(t_0)\mathbf{r}_{\text{IN}}) \right).\end{aligned}\quad (18)$$

However, when it is a prediction in question, we actually do not know the exact matrix $\mathbf{W}(t)$. However, we might know the polar motion parameters x_p and y_p at the beginning of prediction and because they do not change that much in a couple of days long prediction, we can approximate the matrix by $\mathbf{W}(t_0)$. Similarly the $dUT1$ value at the time t can be approximated by the $dUT1$ value at t_0 while computing the matrix $\mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t)$.

We need to do some transformations between different reference frames not only at the beginning and in the end

of the prediction but also during the prediction. In the section 2 it was mentioned that we need to represent the satellite's position vector in an Earth fixed reference frame to calculate the acceleration due to the Earth's gravitation. TRS is an Earth fixed reference frame, so we can do the transformation to it using the equation (18). Therefore the transformation matrix \mathbf{R} presented in equation (5) is $\mathbf{W}(t)\mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t)\mathbf{R}_{\text{TIRS}}^{\text{CRS}}(t_0)$. Because we need to do this transformation at every time step of numerical integration, it is good to pay attention to the computational complexity of this matrix. To speed up the calculation, we approximate the matrix \mathbf{R} in the integration. If the length of prediction is only some days, the precession and nutation matrices remain almost unchanged. Thus we can write $\mathbf{P}\mathbf{P}^T \approx \mathbf{I}$ and $\mathbf{N}\mathbf{N}^T \approx \mathbf{I}$. The matrix

$$\mathbf{W}(t)\mathbf{G}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{P}^T(t_0)\mathbf{N}^T(t_0)\mathbf{G}^T(t_0)$$

can be approximated by

$$\mathbf{W}(t)\mathbf{G}(t)\mathbf{G}^T(t_0).$$

Multiplication with the matrix \mathbf{G} rotates the reference frame according to the Earth's rotation. Mainly it is a simple rotation around the z -axis, with the angular speed of the Earth. If the x -axis points to certain meridian at the initial time t_0 , then at the time t it points to the direction we get by rotating the x -axis around the z -axis with an angle of $\alpha = (t - t_0)\omega$. Thus

$$\mathbf{W}(t)\mathbf{G}(t)\mathbf{G}^T(t_0) = \mathbf{W}(t)\mathbf{R}_z((t - t_0)\omega)$$

is the matrix \mathbf{R} used to transform the inertial vector to an Earth fixed reference frame when computing the Earth gravitational acceleration. Its transpose is then used to transform the acceleration vector to the IN reference frame. Again, in the prediction the matrix $\mathbf{W}(t)$ is replaced with $\mathbf{W}(t_0)$.

There is still one transformation between the reference frames to be presented. The model that is used to compute the solar and lunar positions gives the coordinates in the J2000.0 ecliptic reference frame, where the z -axis points to the mean pole at epoch J2000.0 and xy -plane is the orbital plane of the Earth. We first do the transformation to the equatorial coordinates or to CRS as follows:

$$\mathbf{r}_{\text{cb}} = \mathbf{R}_x(\varepsilon)\mathbf{r}_{\text{cb}},$$

where $\varepsilon = 23.43929111^\circ$ is the obliquity of the ecliptic. Then we transform the coordinates further to the chosen inertial reference frame. Because CRS is also an inertial reference frame, we just do a multiplication with a constant matrix, that is

$$\mathbf{r}_{\text{cb,IN}} = \mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t_0)\mathbf{r}_{\text{cb,CRS}}.$$

After this the coordinate vector $\mathbf{r}_{\text{cb,IN}}$ of a celestial body is ready to be substituted in the equation (7) or (8).

4 INITIAL VALUE IMPROVEMENT

In the previous sections we have presented how to predict the satellite's orbit by solving its equation of motion using satellite's positions and velocity at t_0 as initial conditions. In addition we need to know two of the Earth orientation parameters: x_p and y_p in order to be able to calculate the needed transformation matrix $\mathbf{W}(t_0)$. The third Earth orientation parameter, dUT1, is also needed when computing the matrices $\mathbf{R}_{\text{CRS}}^{\text{TIRS}}(t)$, but we set it zero when doing the prediction. Because the parameters x_p and y_p are unpredictable, we have to get them somehow from the broadcast ephemeris, as we get the initial position and velocity. Otherwise this algorithm would not be suitable for a totally self-assisted GPS device. We now call the satellite's initial state and polar motion parameters as initial conditions, and present how the values of these initial conditions are set up.

The GPS satellites's position can be calculated using the 16 ephemeris parameters that are broadcast by the satellite. For further information of the computation the satellite's position see [13] or [14]. The received position is in WGS84 but we assume it to be in TRS system, which is very close to WGS84. By differentiating the parameters with respect to time, we can also calculate satellite's velocity in TRS, as described in [15]. Also an instructional implementation for this is available in [16].

Having implemented the equations of motion described in Section 2, we saw our model was operating very well when using precise ephemeris position and velocity as a initial condition. Unfortunately the broadcast position and velocity are more inaccurate. Moreover, a small error in the satellite's initial state causes a large error in the final predicted position. Therefore it is necessary to find means to improve the satellite's initial state.

The first thing we do to improve the initial state accuracy is to add an antenna correction to the broadcast ephemeris position. By antenna correction we mean the difference between the satellite's center of mass and antenna phase center. When positioning with GPS, the receiver measures the pseudoranges between the satellites and itself using the signal transmitted from satellites antennas. Thus the broadcast ephemeris position describes the position of satellite's antenna. However, when doing the orbit prediction, it is physically more correct to start the prediction from the satellite's center of mass. We have provided antenna offsets used by NGA, which are documented in [17]. For IIA satellites the offset is

$$\boldsymbol{\delta} = [0.2794 \text{ m} \quad 0.0000 \text{ m} \quad 0.9519 \text{ m}]^T,$$

and for block IIR the offsets are satellite-specific. The offset vectors $\boldsymbol{\delta}$ are given in a satellite body fixed reference frame. We now let \mathbf{r} be the satellite's position in an Earth

centered reference frame, for example TRS and let \mathbf{e}_{sun} be a unit vector from satellite to sun in the same reference frame. Then the unit vectors pointing to the x , y and z directions of satellite body fixed frame can be written as

$$\mathbf{u}_z = -\frac{\mathbf{r}}{\|\mathbf{r}\|}, \mathbf{u}_y = -\frac{\mathbf{e}_{\text{sun}} \times \mathbf{u}_z}{\|\mathbf{e}_{\text{sun}} \times \mathbf{u}_z\|} \text{ and } \mathbf{u}_x = \frac{\mathbf{u}_y \times \mathbf{u}_z}{\|\mathbf{u}_y \times \mathbf{u}_z\|}$$

in TRS. The unit vector \mathbf{u}_z is pointing towards the Earth, \mathbf{u}_y is chosen such that it is perpendicular to the plane containing both the Earth and the Sun, and \mathbf{u}_x defined such that it completes the right-handed system. Using these, the antenna offset in TRS can be written as

$$\boldsymbol{\delta} = \delta_x \mathbf{u}_x + \delta_y \mathbf{u}_y + \delta_z \mathbf{u}_z.$$

The antenna offset vector $\boldsymbol{\delta}$ gives the antenna's position coordinates with respect to the center of mass of the satellite. Now we know the coordinates of antenna phase center, that is the satellite's broadcast position \mathbf{r} and want to calculate the center of mass coordinates \mathbf{r}_{com} . It can be done by subtracting the offset:

$$\mathbf{r}_{\text{com}} = \mathbf{r} - \boldsymbol{\delta}.$$

The opposite correction, from center of mass to antenna phase center, could be done after the prediction. However, in our tests we do not do it, because we compare the predicted satellite positions to the precise ephemeris of NGA [10], which is given in terms of the center of mass.

After applying the antenna correction, we solve the unknown polar motion parameters and improve the inaccurate BE velocity by fitting these variables to the broadcast data. The fitting procedure goes along as described in the following and the Figure 2 illustrates the used notations.

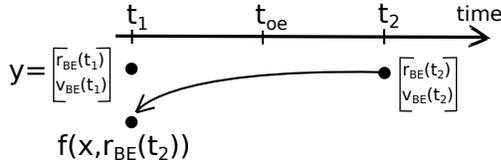


Figure 2: Least squares fitting. We solve the initial velocities of n satellites and polar motion parameters, which form the variable \mathbf{x} , such that the prediction function \mathbf{f} would predict the satellite track going through the two satellite states around the TOE = t_{oe} . That is, we want to find the \mathbf{x} that minimizes the distance between $\mathbf{f}(\mathbf{x}, \mathbf{r}_{\text{BE}}(t_2))$ and \mathbf{y} .

Consider one ephemeris parameter set, which is received as a broadcast message and having a certain time of ephemeris (TOE). With these parameters we are able to calculate the satellite's state i.e. position and velocity at any time within ± 2 h from the TOE. Going outside of this range the precision of BE deteriorates rapidly. We now compute satellite's state at the time instants $t_1 = \text{TOE} - 1.5$ h and $t_2 = \text{TOE} + 1.5$ h for n satellites, from which we have received the broadcast information. Let us denote \mathbf{y} the vector containing the antenna corrected BE positions and

the BE velocities of all these satellites at the first time instant t_1 i.e.

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_r \\ \mathbf{y}_v \end{bmatrix},$$

where

$$\mathbf{y}_r = \begin{bmatrix} \mathbf{r}_{\text{BE}}^{\text{sat1}}(t_1) \\ \vdots \\ \mathbf{r}_{\text{BE}}^{\text{sat}n}(t_1) \end{bmatrix} \text{ and } \mathbf{y}_v = \begin{bmatrix} \mathbf{v}_{\text{BE}}^{\text{sat1}}(t_1) \\ \vdots \\ \mathbf{v}_{\text{BE}}^{\text{sat}n}(t_1) \end{bmatrix}.$$

Then we denote \mathbf{f} the function which, starting from latter time instant t_2 , computes the satellites' states at t_1 . This function carries out first a coordinate transformation from Earth fixed TRS to the inertial reference frame IN according to the eq. (14) and (17), secondly it solves a nonlinear differential equation by evaluating the integral (2) and finally does the transformation back to TRS using (18) and (19). As input parameters this function requires satellites' positions and velocities at t_2 as well as the polar motion parameters x_p and y_p of the same time instant. Polar motion parameters are needed when calculating the transformation matrices and they are common to all satellites. The third of the EOP:s, dUT1, is also unknown, but we set it zero when evaluating the value of function \mathbf{f} . We fix the initial positions of the satellites on the antenna corrected BE positions denoted $\mathbf{r}_{\text{BE}}(t_2)$. The rest of the function inputs form the variable vector

$$\mathbf{x} = \begin{bmatrix} x_p \\ y_p \\ \mathbf{v}_{\text{BE}}(t_2) \end{bmatrix},$$

where

$$\mathbf{v}_{\text{BE}}(t_2) = \begin{bmatrix} \mathbf{v}_{\text{BE}}^{\text{sat1}}(t_2) \\ \vdots \\ \mathbf{v}_{\text{BE}}^{\text{sat}n}(t_2) \end{bmatrix}.$$

Now we want to find the $\hat{\mathbf{x}}$, with which the value of the function \mathbf{f} is as near the point \mathbf{y} as possible. In other words we solve the nonlinear least squares fitting problem

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \left(\sum_i p_i^2(\mathbf{x}) \right) = \arg \min_{\mathbf{x}} \mathbf{p}^T(\mathbf{x}) \mathbf{p}(\mathbf{x})$$

where the residual function \mathbf{p} is

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} \mathbf{f}_r(\mathbf{x}, \mathbf{r}_{\text{BE}}(t_2)) - \mathbf{y}_r \\ (\mathbf{f}_v(\mathbf{x}, \mathbf{r}_{\text{BE}}(t_2)) - \mathbf{y}_v) \cdot 1000 \end{bmatrix}.$$

Here we have partitioned the vector valued function \mathbf{f} into two pieces, such that the first contains all the components representing the satellites' positions and the second contains the velocities. The magnitude of the velocity residuals differs significantly from the position residuals. We have found experimentally, that multiplying the velocity residual by 1000 works well in this case. Instead of the residual function we could have put the

weights also to the function to be optimized ending up to the weighted least squares problem

$$\hat{\mathbf{x}} = \arg \min_x (\mathbf{p}^T(\mathbf{x})\mathbf{D}\mathbf{p}(\mathbf{x})),$$

where the diagonal matrix \mathbf{D} has a value of $\sqrt{1000}$ in those elements corresponding to the velocity components and ones elsewhere.

The nonlinear least squares problem can be solved with Levenberg-Marquardt method. The algorithm requires an initial guess for \mathbf{x} . For the velocities $\mathbf{v}(t_2)$ the initial guess is taken from the broadcast velocity $\mathbf{v}_{BE}(t_2)$. For polar motion parameters we have used the values $x_{p0} = 0.05$ arcseconds and $y_{p0} = 0.35$ arcseconds, which is the approximate center of the polar motion spiral during the years 2004-2008. In practice the initial guess for polar motion could be taken from the results of previous LSQ-solution. An initial guess near the true value speeds up the convergence of the least squares fitting algorithms.

As a result we get quite accurate polar motion parameters \hat{x}_p and \hat{y}_p as well as improved velocities of the satellites $\hat{\mathbf{v}}$ at the time instant t_2 . Thus we are able to start our prediction from t_2 to the future.

5 TESTS

We tested our algorithm with broadcast ephemeris from GPS weeks 1540-1580 and having the TOE equal to 57600 s. The initial conditions were fitted with the ephemeris of five satellites at time, after which predictions were done for each of these satellites. There was about 160 fittings and 800 predictions together. The error in the predicted satellite position was computed by taking the norm of the difference between predicted position and NGA precise ephemeris position. We chose the 95% quantile of these errors and this is plotted as a blue line in Figure 3. Furthermore, green line is the corresponding error quantile, if we could use NGA precise ephemeris and the exact EOP as initial conditions in our force model.

When considering the results of the Figure 3, one should bear in mind that the presented satellite's position error can not be converted directly to the final positioning error of the user. The error component having strongest effect to the final positioning error is the error component parallel to the line between the user and the satellite, and this is varying together with the user's location on the Earth. However, the error curve of the Figure 3 gives a good upper bound to this user range error (URE). Another factor that has a significant effect on the positioning error, is the clock offset of the satellite, which is unpredictable. We do not consider this error here but it is discussed in [2].

To illustrate the influence of the initial value improvement algorithm of Section 4 we plotted a prediction error,

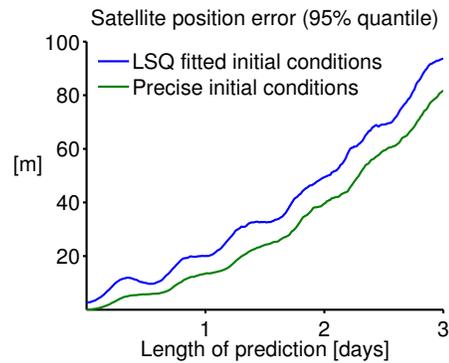


Figure 3: Error in satellite's position as a function of prediction length. Blue line is the algorithm, which could be running in an autonomous positioning device. Green line is the prediction error with the same force model, if we could start the prediction from satellite's precise ephemeris state and knew the EOP at the beginning of prediction.

where the initial conditions were taken directly from the broadcast and another line for case where we did the antenna correction to the broadcast position, but no fitting. These curves are shown in the Figure 4. Comparing to Figure 3 we can see that fitting the initial velocity with this least squares algorithm has a great impact to the prediction error.

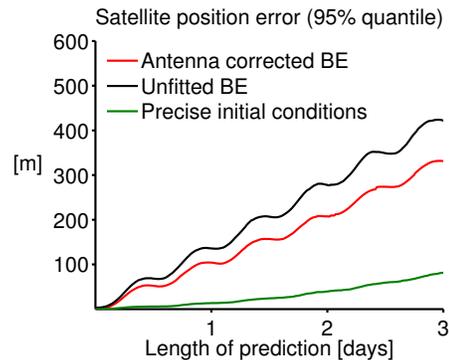


Figure 4: How the initial conditions affect to the prediction result? Here our simple force model is tested with initial conditions taken from a) broadcast directly (black line) b) antenna corrected broadcast (red line) c) NGA precise ephemeris (green line, same as in Figure 3). In every case the EOP is assumed to be known at the beginning of prediction.

In this paper we simulated the presented least squares fitting algorithm using five satellites' broadcasts, which all had the same TOE. Note that identical TOEs were assumed for simplicity and that the algorithm can equally well handle broadcasts having different TOEs. Indeed, this is an important notion as in real applications the TOEs differ from broadcast to broadcast, depending on what the receiver has collected. However one should not use too old, say over a week old broadcasts. This is because EOP are changing with respect to time and this is not taken into account in our algorithm.

6 CONCLUSION

We presented a method to predict satellite tracks in an autonomous navigation device, which works without needing any network connection. Broadcast ephemeris is the only data it receives. The presented method is suited to reduce TTFF when the Assisted GPS is not available.

Our method for predicting satellite's position is based on solving satellite's equation of motion numerically. We have included only the most significant forces: Earth gravitation, solar and lunar gravitation and solar radiation pressure to the equation of motion, which we call the force model.

After showing how to form the satellite's equation of motion we presented a method to solve satellites' initial positions and velocities from the broadcast ephemeris, such that they will suit better as initial conditions of the force model. We first do an antenna correction to the broadcast position, after which we use these corrected satellite positions together with our force model to solve the velocities of the satellites as well as the polar motion parameters, which are needed in order to know the exact rotation axis of the Earth.

We tested our method with GPS broadcast ephemeris data and calculated the prediction error using the precise ephemeris delivered by NGA as a reference. In our test set 95% of the prediction errors were below 21 meters for one day prediction and below 94 meters for three day prediction. Noting that in three days the satellite has travelled about 1 000 000 km, this level of accuracy is notable. To appreciate the benefit of the presented least squares fitting method we also calculated the prediction error of our force model, when the initial state was taken a) from precise ephemeris and b) from pure or unfitted broadcast ephemeris. With precise ephemeris as initial condition, the force model is almost order of magnitude better than with pure broadcast. However, with the presented fitting method the prediction accuracy with BE got very close to the accuracy of predictions with PE. In addition, the least squares fitting method can solve the polar motion parameters, which are likely to be unknown for the device.

We believe that error in an initial state that is based on precise ephemeris has a negligible effect on satellite orbit prediction, compared to errors in the force model. Furthermore, because the prediction accuracy with fitted initial conditions is very close to this, we assume that also the most significant error source in that case arises from the force model.

For radiation pressure we used a simple Cannonball-model and the magnitude of the force was estimated separately for each satellite. However, when GPS tracks are predicted with high level accuracy, the used radiation pressure

models are more sophisticated and the free parameters are estimated just before starting the prediction, using precise ephemeris data. In our case the device has received only some broadcast ephemeris sets, whose accuracy is worse and therefore this kind of estimation might not be possible. Therefore we believe the model for solar radiation pressure is constricting the level of accuracy. To enhance the accuracy one should, if possible, try to make the model for solar radiation pressure better.

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