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Bayesian Fault Detection Method for Linear Systems with Outliers

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***Abstract*—A novel approach for monitoring the accuracy of the Bayesian estimate of linear Gaussian state-space model is introduced, based on the monitoring of the propagation of the errors in the Kalman filter algorithm. The effect of the sensor errors on the Kalman filter estimate is explicitly computed and compensated for. Marginalized particle filter is used to compute the posterior distribution of the sensor errors and using a target tracking simulation it is shown that the proposed method has improved performance over the standard DIA method**

***Index Terms*—Bayesian filtering, marginalized particle filtering, fault diagnosis, jump detection, change detection, fault monitoring, Kalman filter, DIA**

I. Introduction

Abrupt changes in linear dynamic systems are often of significant interest as they can provide essential information about the processes, or possibly cause major degeneracy of the state estimator if the changes of the system go undetected. For example, in clinical trials, the changes in the system can be caused by biological events which are of paramount importance to analyze [1]. In positioning and tracking systems, the changes in the environment or maneuvers cause the system to provide biased position estimates which can lead to hazardous situations [2].

In the positioning systems, the fault detection methods are usually referred to as receiver autonomous integrity monitoring (RAIM) methods [3]. The traditional RAIM methods first perform fault detection based on a statistical test for the consistency of the observation vector. If the test fails, statistical tests are performed on each of the observations in order to identify and remove the faulty observation [4]. The diagnosis is carried out at each time step by testing the bias of each of the observations separately but there has been studies of integrity and quality monitoring methods in time series data [5], [6], [7].

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We model the abrupt changes, or errors, in the system as suddenly appearing or disappearing additive components in the sensor model. In positioning systems, cause of these kind of errors could be e.g. multipath or non-line-of-sight-signals, or sensor malfunctions [8]. The detection of these kind of changes have traditionally been performed with generalized likelihood ratio (GLR), or almost analogous detection-identification-adaptation (DIA) method, and CUSUM algorithms [9], [5], [10], [11]. These methods are based on monitoring the innovation process of a Kalman filter (KF) that does not take into account the abrupt changes. Another approach would be to approximate the joint posterior distribution of the state and the abrupt changes using multiple model filtering [12], or sequential Monte Carlo methods [13], [2]. From the resulting posterior distribution one can solve for any quality measure of the chosen estimator. However, quality measures based on the posterior distribution can be quite sensitive to the probabilities on the tails of the posterior which can be poorly estimated by sampling based methods.

We propose a change detection method which combines the two approaches. Instead of solving the joint distribution of the state and the changes, we compute the joint distribution of the changes and the KF estimate error caused by the additive changes. One benefit of this approach is that the detection procedure is separate from the state estimator and can be applied as a separate module to any system estimated by KF.

The paper is organized as follows. In Section II we describe the state-space model for the system and for the additive sensor errors. In Section III we present a Bayesian approximative solution for the problem which employes marginalized particle filter, which can be used due to the special property of the proposed model. In Section IV we describe our novel approach for fault diagnosis of the nominal Kalman filter and in Section V we compare the presented approaches in a simple positioning problem. In Section VI we conclude our study.

II. Problem formulation

We consider a discrete time stochastic system with additive unknown changes

$$x_{k+1} = F_k x_k + w_k \tag{1}$$
$$y_k = H_k x_k + v_k + s_k. \tag{2}$$
$$x_0 \sim \mathsf{N}\left(x_{0|0}, P_{0|0}\right), \tag{3}$$

where $x_k \in \mathsf{R}^{n_x}$ is the state vector, $y_k \in \mathsf{R}^{n_y}$ is the observation, $w_k \sim \mathsf{N}(0, Q_k)$ and $v_k \sim \mathsf{N}(0, R_k)$ are mutually independent, zero mean Gaussian noise processes. s_k is the additional error process in the sensor model. $\mathsf{N}(\mu, \Sigma)$ is a Gaussian distribution with mean μ and covariance Σ.

Bayesian filtering framework can be used to compute the posterior distribution $p(x_k \mid y_{1:k}, s_{1:k})$, where $y_{1:k} \triangleq [y_1, \ldots, y_k]$, with the *prediction* step

$$p(x_k \mid y_{1:k-1}, s_{1:k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid y_{1:k-1}, s_{1:k-1}) \mathrm{d}x_{k-1} \tag{4}$$

and the *update* step

$$p(x_k \mid y_{1:k}, s_{1:k}) \propto p(y_k \mid x_k, s_k) p(x_k \mid y_{1:k-1}, s_{1:k-1}). \tag{5}$$

It is well known that with the given model (1) – (3) and known $s_{1:k}$ the posterior distribution is a Gaussian distribution $p(x_k | y_{1:k}, s_{1:k}) = \mathsf{N}\left(x_{k|k}, P_{k|k}\right)$ with mean and covariance computed recursively by the KF algorithm

$$x_{k+1|k} = F_k x_{k|k} \tag{6}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + Q_k \tag{7}$$
$$z_k = y_k - H_k x_{k|k-1} - s_k \tag{8}$$
$$S_k = H_k P_{k|k-1} H_k^T + R_k \tag{9}$$
$$K_k = P_{k|k-1} H_k^T S_k^{-1} \tag{10}$$
$$x_{k|k} = x_{k|k-1} + K_k z_k \tag{11}$$
$$P_{k|k} = (I_{n_x} - K_k H_k) P_{k|k-1}, \tag{12}$$

where I_n is a $n \times n$ identity matrix, z_k is the innovation and S_k is the innovation covariance. The additive errors s_k are usually unknown and, as they have a linear (and hence unbounded) influence on the state estimates, they can significantly degrade the performance of KF algorithm.

We model the additive errors s_k as a Gaussian Markov process depending on a Markov chain $\lambda_k \in \{0, 1\}$ with

$$\mathsf{P}(\lambda_{k+1} = j \mid \lambda_k = i) = p_{ij} \tag{13}$$

which is the switch probability between ith and jth models at time $k+1$. Modeling of p_{ij} can be often an extremely complicated task, as it can be dependent on the time k, and also on the value of x_k [14].

Also the size of the errors is a difficult to model as the cause of the errors is often unknown, and assuming the value of the sensor error to be e.g. constant in time, is quite a strong assumption. Therefore we consider additive errors as jump Markov linear system

$$s_{k,i} = \lambda_{k,i} \epsilon_{k,i}, \tag{14}$$

where $s_{k,i}$ is the ith element of s_k at kth timestep and $\epsilon_k \sim \mathsf{N}(0, R_k^\epsilon)$ is Gaussian white noise process independent of the stochastic processes in (1)–(3). Using this model we estimate the value of $e_{k,i}$ independent of the estimated $e_{k-1,i}$. Note that

$$v_k + s_k \sim \mathsf{N}(0, R_k + \Lambda_k R_k^\epsilon \Lambda_k) = \mathsf{N}(0, R_k(\Lambda_k)), \tag{15}$$

where $\Lambda_k \triangleq \mathrm{diag}(\lambda_{k,1}, \ldots, \lambda_{k,n_y})$. This model is often used to describe outliers in the observations [15] and it is, in a sense, a conservative model for the sensor errors. Often errors are modeled as a constant, or slowly evolving, bias for consecutive time steps [2]. However, if nothing is known about the dynamic nature of the error, the assumption of constant bias may degrade the performance of the estimator significantly. On the other hand, if we assume that the size of the bias may change freely from a time step to the next, we may be able to estimate it better in the case when it is not constant or slowly evolving.

III. Marginalized particle filtering

The posterior distribution for the model introduced in the previous section is

$$p(x_k, \Lambda_{1:k} \,|\, y_{1:k}) = p(\Lambda_{1:k} \mid y_{1:k}) p(x_k \mid y_{1:k}, \Lambda_{1:k}). \tag{16}$$

The indicator variable history $\Lambda_{1:k}$ is a discrete random variable with a finite number (n^{y_k}) of possible values and probability mass function

$$p(\Lambda_{1:k} \mid y_{1:k}) = \sum_{i=1}^{n_y^k} \mathsf{P}\left(\Lambda_{1:k}^{(i)} \mid y_{1:k}\right) \delta\left(\Lambda_{1:k} - \Lambda_{1:k}^{(i)}\right),$$

where $\mathsf{P}\left(\Lambda_{1:k}^{(i)} \mid y_{1:k}\right)$ is a shorthand notation for the probability $\mathsf{P}\left(\Lambda_{1:k} = \Lambda_{1:k}^{(i)} \mid y_{1:k}\right)$. The marginal distribution of the state is

$$p(x_k \mid y_{1:k}) = \sum_{i=1}^{n_y^k} \mathsf{P}\left(\Lambda_{1:k}^{(i)} \mid y_{1:k}\right) p\left(x_k \mid y_{1:k}, \Lambda_{1:k}^{(i)}\right),$$

which can be computed with a bank of KFs [16], [17]. The sum goes over all possible $\Lambda_{1:k}$ and thus the exact

solution is computationally intractable for even small k. Several approximative techniques have been applied for this problem, e.g. pruning or merging the Gaussian components [12], [18].

We apply sequential Monte Carlo estimation for approximating the posterior [19]. Because the part $p(x_k|y_{1:k}, \Lambda_{1:k})$ of the posterior can be solved analytically, it is possible to approximate only the marginal distribution $p(\Lambda_{1:k} \mid y_{1:k})$ and thus decrease the variance of the empirical approximative posterior [13]. The resulting estimation method is the marginalized particle filter (MPF). Here are the details.

We approximate $p(\Lambda_{1:k}|y_{1:k})$ empirically with N samples

$$p(\Lambda_{1:k} \mid y_{1:k}) \approx \sum_{j=1}^{N} \omega_{1:k}^{(j)} \delta\left(\Lambda_{1:k} - \Lambda_{1:k}^{(j)}\right), \tag{17}$$

where

$$\omega_{1:k}^{(j)} \propto \frac{\mathsf{P}\left(\Lambda_{1:k}^{(j)} \mid y_{1:k}\right)}{\pi\left(\Lambda_{1:k}^{(j)} \mid y_{1:k}\right)}. \tag{18}$$

The importance sampling distribution $\pi\left(\Lambda_{1:k} \mid y_{1:k}\right)$ can be chosen freely within certain requirements but the choice

$$\pi(\Lambda_{1:k} \mid y_{1:k}) = \pi\left(\Lambda_1\right) \prod_{j=1}^{k} \pi\left(\Lambda_j \mid \Lambda_{j-1}\right), \tag{19}$$

enables recursive updating of weights $\omega_{1:k}$ according to

$$\omega_{1:k}^{(j)} \propto \omega_{1:k-1}^{(j)} p\left(y_k \mid y_{1:k-1}, \Lambda_{1:k}^{(j)}\right). \tag{20}$$

The latter term is evaluated by sampling $\Lambda_k^{(j)}$ from (19), adding it to $\Lambda_{1:k-1}^{(j)}$ and evaluating

$$\begin{aligned} & p\left(y_k \mid y_{1:k-1}, \Lambda_{1:k}^{(j)}\right) \\ &= \int p\left(y_k \mid x_k, \Lambda_k^{(j)}\right) p\left(x_k \mid y_{1:k-1}, \Lambda_{1:k}^{(j)}\right) \mathrm{d}x_k \\ &= \frac{1}{\sqrt{\det(2\pi S_k(\Lambda_{1:k}^{(j)}))}} e^{-\frac{1}{2} z_k(\Lambda_{1:k}^{(j)})^T S_k(\Lambda_{1:k}^{(j)}) z_k(\Lambda_{1:k}^{(j)})}, \end{aligned} \tag{21}$$

where $z_k(\Lambda_{1:k}^{(j)})$ and $S_k(\Lambda_{1:k}^{(j)})$ are the innovation and innovation covariance given the history $\Lambda_{1:k}^{(j)}$ and are computed in (8) and (9).

The approximative posterior distribution is now

$$\hat{p}(x_k \mid y_{1:k}) = \sum_{j=1}^{N} \omega_{1:k}^{(j)} \mathsf{N}\left(x_{k|k}(\Lambda_{1:k}^{(j)}), P_{k|k}(\Lambda_{1:k}^{(j)})\right), \tag{22}$$

where (11) and (12) are computed given $\Lambda_{1:k}^{(j)}$. In practice we need to occasionally *resample* the Gaussian mixture components due to the degeneracy of the weights of the approximative distribution. In the resampling procedure, the components with large weights are duplicated and used to replace components with small weights if the effective sample size N_{eff} drops lower than some threshold value [19]. The effective sample size can be approximated as

$$N_{\text{eff}} \approx \frac{1}{\sum_{i=1}^{N} \left(\omega_{1:k}^{(i)}\right)^2}. \tag{23}$$

IV. Nominal system fault diagnosis

Many of the classic change detection and quality monitoring algorithms, such as GLR method [9] and detection-identification-adaptation (DIA) method [5], are based on the innovation of the nominal KF. The nominal KF is run with the assumption that $\Lambda_i = 0$ for all $i \geq 1$. Due to the recursive nature of the KF algorithm, the error propagates according to Lemma 1.

Lemma 1. *Let the state space model be described by* (1)–(3). *The influence of the realized additive error sequence $s_{1:k}$ on the Kalman innovation* (8) *and the posterior mean* (11) *can be expressed explicitly as*

$$z_k = z_k(0_{1:k}) + \Delta z_k \tag{24}$$

and

$$x_{k|k} = x_{k|k}(0_{1:k}) + \Delta x_{k|k} \tag{25}$$

where

$$z_k^0 = y_k - H_k x_{k|k-1}(0_{1:k}) \tag{26}$$

The sequences Δz_k and $\Delta x_{k|k}$ can be expressed recursively as

$$\Delta z_k = s_k - H_k F_{k-1} \Delta x_{k-1|k-1} \tag{27}$$

and

$$\Delta x_{k|k} = K_k s_k + C_k \Delta x_{k-1|k-1}, \tag{28}$$

where $C_k = (I_{n_x} - K_k H_k) F_{k-1}$.

Proof: Analogous to the proof of Lemma 5 in [20]. ■

Instead of solving the marginal distribution of the state, we compute the posterior distribution of the additive errors and the error of the nominal KF estimator. Lemma 1 describes the evolution of the influence of the additive errors on the state, the innovation, and the KF estimator. Using the lemma, the quality monitoring procedure is formulated as a linear system with white

noise processes as process uncertainty. The system is observed through (24).

$$\begin{bmatrix} s_{k+1} \\ \Delta x_{k|k} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ K_k & C_k \end{bmatrix} \begin{bmatrix} s_k \\ \Delta x_{k-1|k-1} \end{bmatrix} + \begin{bmatrix} \Lambda_{k+1}\epsilon_{k+1} \\ 0 \end{bmatrix} \tag{29}$$

$$z_k = \begin{bmatrix} I_{n_y} & -H_k F_{k-1} \end{bmatrix} \begin{bmatrix} s_k \\ \Delta x_{k-1|k-1} \end{bmatrix} + z_k^0, \tag{30}$$

$$\begin{bmatrix} s_0 \\ \Delta x_{0|0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

where $z_k(0_{1:k}) \sim N(0, S_k(0_{1:k}))$ is a white noise process independent of s_k and $\Delta x_{k|k}$. We can solve the system (29) – (31) in the Bayesian framework as

$$\begin{aligned} &p(s_k, \Delta x_{k|k} \mid z_{1:k}) \\ &= \sum_{j=1}^{n_y^k} \mathsf{P}\left(\Lambda_{1:k}^{(j)} \mid z_{1:k}\right) p(s_k, \Delta x_{k|k} \mid z_{1:k}, \Lambda_{1:k}). \end{aligned} \tag{32}$$

The approximative distribution is computed using MPF analogously to the previous section. The resulting distribution is

$$\begin{aligned} &\hat{p}(s_k, \Delta x_{k|k} \mid z_{1:k}) \\ &= \sum_{j=1}^{N} \omega_{1:k}^{(j)} \mathsf{N}\left(\begin{bmatrix} s_{k|k}(\Lambda_{1:k}^{(j)}) \\ \Delta x_{k|k}(\Lambda_{1:k}^{(j)}) \end{bmatrix}, \Sigma_{k|k}(\Lambda_{1:k}^{(j)})\right), \end{aligned} \tag{33}$$

where the mean and the covariance are computed applying the KF algorithm to the linear Gaussian system (29) – (31) given $\Lambda_{1:k}^{(j)}$. It can be shown that given $\Lambda_{1:k}$ the weights for the approximative distributions of the previous section and of the introduced model are the same, i.e.

$$\mathsf{P}\left(\Lambda_{1:k} \mid y_{1:k}\right) = \mathsf{P}\left(\Lambda_{1:k} \mid z_{1:k}\right). \tag{34}$$

The posterior filtering distribution $\mathsf{P}\left(\Lambda_k \mid z_{1:k}\right)$ is the probability of an error being present in the observation, and can be used to determine the quality of the observation vector. If $\mathsf{P}\left(\lambda_{k,i} = 1 \mid z_{1:k}\right) > 0.5$, then it is more probable that the error is present than not, given the whole observation history. In addition, to determining whether an error is present, we are able to monitor the size of the cumulative effect $\Delta x_{k|k}$ of the sensor errors $s_{1:k}$. We use the mean of $\Delta x_{k|k}$ as the estimate of the sensor error size, and using this estimate we can compute a corrected estimate $\tilde{x}_{k|k}$ using the filter estimate $x_{k|k}$ with the estimated error $\Delta x_{k|k}$

$$\tilde{x}_{k|k} = x_{k|k} - \Delta x_{k|k}. \tag{35}$$

The quality monitoring method is illustrated by the Figure 1.

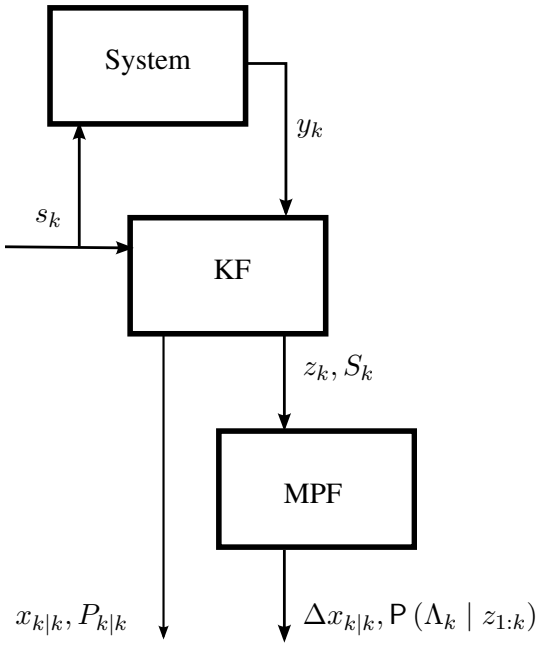

Fig. 1. Quality monitoring of the nominal KF

V. Simulations

We test the discussed estimation method, referred to as nominal system fault detection (NSFD) in a simulation of a simple two-dimensional positioning tracking problem in urban environment. At each time step we receive the position coordinates of the receiver as the measurement. The state consists of two position and two velocity coordinates, and the motion of the target (1) is modeled with the constant velocity model [12]

$$F_k = \begin{bmatrix} I_2 & I_2 \\ 0_2 & I_2 \end{bmatrix}, \quad Q_k = 0.1^2 \begin{bmatrix} \frac{1}{3}I_2 & \frac{1}{2}I_2 \\ \frac{1}{2}I_2 & I_2 \end{bmatrix}. \tag{36}$$

The nominal measurement model is (2) is

$$H_k = \begin{bmatrix} I_2 & 0_2 \end{bmatrix}, \quad R_k = \begin{bmatrix} 7^2 & 3^2 \\ 3^2 & 8^2 \end{bmatrix}. \tag{37}$$

We simulate 1000 tracks with 300 time steps. In time interval $k \in [101, 200]$ we simulate outliers in the additive sensor error with model (14), using $p_{00} = p_{11} = 0.9$ and $\epsilon_k \sim \mathsf{N}\left(0, 30^2 I_2\right)$. We run the MPFs with $N = 25$ particles, and use effective sample size threshold $0.6 \cdot N$ in the resampling procedure. The mean value of the approximative posterior distribution is used as the estimate. DIA method that tests the presence of bias at each time step with test statistic threshold $T_{\mathrm{DIA}} = 5$ is used as a comparison.

The simulation results are illustrated with Table I. The correlation coefficient of the true error of the nominal KF and the estimated error $\Delta x_{k|k}$ given by NSFD is 0.77. Using the estimated $\Delta x_{k|k}$ to evaluate a new state

estimate $\tilde{x}_{k|k}$ in (35), the RMSE of the KF 5.43 is lowered to 4.38. This is an improvement to the RMSE 5.11 of DIA method. The errors are computed in the two-dimensional position coordinates, i.e. the first two dimensions of the state vector.

The failure detection power of NSFD and DIA is compared by the ability to detect presence of the bias in observations. This is quantified with the frequency of false positives (type I error), i.e. determining that there is an error present when there is not, and false negatives (type II error), i.e. determining that there is no bias, when in reality there is. NSFD has greater than DIA with respect to both type I and II errors. However, although NSFD has improved performance of DIA with respect to error detection power and RMSE performance, NSFD is computationally more demanding. The current implementation of NSFD takes roughly N times more time than DIA, where N is the number of particles in (33).

	NSFD	DIA	KF
Type I error	0.04	0.11	
Type II error	0.18	0.26	
RMSE	4.38	5.11	5.43

TABLE I
PERFORMANCE COMPARISON OF ERROR DETECTION POWER AND RMSE OF NSFD AND DIA METHODS.

VI. CONCLUSIONS

A new novel fault detection approach NSFD has been proposed for linear Gaussian state-space models, and marginalized particle filtering solution has been provided for solving the resulting problem. The method was tested against standard DIA method using positioning simulations, and NSFD has better performance with respect to the fault detection power and RMSE. However, the improved performance is achieved with the cost of more demanding computational requirements.

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