TAMPEREEN TEKNILLINEN YLIOPISTO
TAMPERE UNIVERSITY OF TECHNOLOGY
Julkaisu 707 • Publication 707

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On the Stability of the Discrete Time Filter and the Uniform Convergence of Its Approximations

Tampere 2007

Tampereen teknillinen yliopisto. Julkaisu 707
Tampere University of Technology. Publication 707

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Thesis for the degree of Doctor of Technology to be presented with due permission for public examination and criticism in Rakennustalo Building, Auditorium RG202, at Tampere University of Technology, on the 14th of December 2007, at 12 noon.

Tampereen teknillinen yliopisto - Tampere University of Technology
Tampere 2007

ISBN 978-952-15-1892-8 (printed)
ISBN 978-952-15-1923-9 (PDF)
ISSN 1459-2045

Abstract

The stochastic discrete time filter also known as the Bayesian or optimal filter has a wide range of applications in modern technology. In general, the filter recursion is intractable and therefore in practice, it has to be approximated by some numerical method. Typically, the accuracy of the approximation can be increased only by allowing the evaluation of the approximation to become computationally more expensive. Moreover, in some applications the filter is run for an indefinite time. In such a case it is desired that with a fixed computational cost the error of the approximation is guaranteed to remain below some level and that this level can be made as small as desired by letting the computational cost of the approximating algorithm increase. If this can be done, then the approximation is said to be uniformly convergent.

It has been pointed out by several authors that the ability to approximate the filter in a uniformly convergent manner is closely related to the stability of the exact filter with respect to its initial conditions. This relation is manifested in the literature by results stating that under some assumptions a stable filter admits uniformly convergent approximations. This thesis addresses both of the above mentioned problems, the stability of the discrete time filter with respect to its initial conditions and the uniform convergence of certain filter approximations.

The main result regarding the stability establishes easily verifiable sufficient conditions for the filter stability. The stability in this case means that the total variation distance between two filters with different initial distributions converges to zero almost surely. Also rates for the convergence are provided by the analysis. Similarly, the main result regarding the uniform convergence establishes sufficient conditions for the uniform convergence of certain filter approximation algorithms. The uniform convergence is proved in the mean sense, not almost surely. Although the stability of the filter is not shown to be one of the sufficient conditions for the uniform convergence, it is shown that similar conditions are sufficient for both, the stability and the uniform convergence.

Perhaps the most important conclusion of the stability theorem is that under some assumptions the filter is shown to be stable provided that the tails of the observation noise distributions are sufficiently light compared to the tails of the signal noise distributions. In particular, the signal is not required to be ergodic or mixing and the state space is not required to be compact. Moreover, it is not assumed in general that the observation noise enters the filter framework with a sufficiently small coefficient. This is assumed only if the observation noise and the signal noise distributions have equally light tails.

The uniform convergence is proved for a general class of filter approximation methods and in particular, it is shown that the conditions are satisfied by a certain auxiliary particle filter type algorithm and by a certain sampling/importance resampling filter type sequential Monte Carlo algorithm. These algorithms are assumed to employ either the multinomial resampling scheme or the tree based branching algorithm. The uniform convergence is obtained with respect to the sample size but because the computational cost of these filter approximation algorithms is determined by the sample size, the convergence is also uniform with respect to the computational cost. Moreover, the uniform convergence results are illustrated by some computer simulations.

Preface

The research reported in this work was funded by the Tampere Graduate School in Information Science and Engineering and it was carried out it the Department of Mathematics, Tampere University of Technology during the period 2004–2007.

I wish to thank my supervisor Prof. Robert Piché for promoting international collaboration and for making it possible for me to fully concentrate on my research without external distractions, my colleagues Simo Ali-Löytty and Niilo Sirola for enlightening discussions and new perspectives to the problems under consideration, and my pre-examiners Dr François Le Gland and Dr Pierre Del Moral. I also wish to express my gratitude to Dr Dan Crisan for his time, advice, patience, energy and enthusiasm for mathematics. His input to my research is truly invaluable. Above all I wish to thank my family.

Tampere, November 2007
Kari Heine

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Chapter 1

Introduction

Let S be a locally compact, second countable topological space and $\mathscr{S}$ the associated Borel σ-field. It is assumed that there is a probability measure P_0 on $\mathscr{S}$ and transition probabilities $K_i : S \times \mathscr{S} \to [0,1]$, $i \in \mathbb{N}$, such that for all $x \in S$, $K_i(x, \cdot)$ is a probability measure on $\mathscr{S}$ and for all $A \in \mathscr{S}$, $K_i(\cdot, A)$ is measurable. We consider the product space

$$(\Omega_X, \mathscr{F}_X) \triangleq \left(\prod_{i=0}^{\infty} S_i, \bigotimes_{i=0}^{\infty} \mathscr{S}_i \right),$$

where $(S_i, \mathscr{S}_i) \triangleq (S, \mathscr{S})$, $i \geq 0$. Let P_X denote the unique probability measure on the product σ-field $\mathscr{F}_X$ satisfying

$$\mathsf{P}_X(A) = \int_{A_0} \int_{A_1} \cdots \int_{A_n} K_n(x_{n-1}, \mathrm{d}x_n) \cdots K_1(x_0, \mathrm{d}x_1) \mathsf{P}_0(\mathrm{d}x_0)$$

for all rectangles $A = \prod_{i=0}^{\infty} A_i$, such that $A_i \in \mathscr{S}$, $i > 0$, and $A_i = S$, $i > n \in \mathbb{N}$. The existence and uniqueness of this probability measure is an elementary result for which the proof can be found e.g. in [53, Corollary 2, page 165]. Thus we have constructed a probability space $(\Omega_X, \mathscr{F}_X, \mathsf{P}_X)$.

Define $X \triangleq (X_i)_{i \geq 0} \triangleq (X_0, X_1, \ldots)$, where $X_i : \Omega_X \to S$, $i \in \mathbb{Z}_+ \triangleq \mathbb{N} \cup \{0\}$ is the projection of Ω_X to the ith coordinate space, i.e., for all $\omega = (\omega_1, \omega_2, \ldots) \in \Omega_X$, one has $X_i(\omega) = \omega_i$. The resulting stochastic process X is called *the signal process*. It is observed that X is Markovian and for all $A \in \mathscr{S}$ and $i > 0$

$$\mathsf{P}(X_i \in A \mid X_{i-1}) = K_i(X_{i-1}, A),$$

for which the proof can be found, e.g. in [53, Proposition V.2.1.].

We also consider another probability space $(\Omega_V, \mathscr{F}_V, \mathsf{P}_V)$ such that $\Omega_V \triangleq \mathbb{R}^\infty$, $\mathscr{F}_V \triangleq \mathscr{B}(\mathbb{R}^\infty)$, and $\mathsf{P}_V \triangleq \prod_{i=1}^\infty \mathsf{P}_{V_i}$. Above, $\mathscr{B}(\mathbb{R}^\infty)$ denotes the Borel field in $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ generated by the Euclidean topology and P_{V_i}, $i > 0$ is a probability measure on the Borel field $\mathscr{B}(\mathbb{R}^{d_\mathrm{m}})$ in $\mathbb{R}^{d_\mathrm{m}}$, where $d_\mathrm{m} \in \mathbb{N}$. For the existence and uniqueness of P_V, see e.g. [53, Proposition V.1.2.]. We define a stochastic process $Y \triangleq (Y_i)_{i>0} \triangleq (Y_1, Y_2, \ldots)$ such that for all $i > 0$

$$Y_i \triangleq h_i(X_i) + V_i, \tag{1.1}$$

where $h_i : S \to \mathbb{R}^{d_\mathrm{m}}$ is measurable and $V_i : \Omega_V \to \mathbb{R}^{d_\mathrm{m}}$ is the projection of Ω_V to the ith coordinate space $\mathbb{R}^{d_\mathrm{m}}$. The processes Y and $V \triangleq (V_i)_{i>0}$ are referred to as *the observation process* and *the observation noise process*, respectively.

Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathsf{P}})$ be a third probability space which for the moment is left unspecified. Define

$$(\Omega, \mathscr{F}, \mathsf{P}) \triangleq (\Omega_X \times \Omega_V \times \tilde{\Omega}, \mathscr{F}_X \otimes \mathscr{F}_V \otimes \tilde{\mathscr{F}}, \mathsf{P}_X \times \mathsf{P}_V \times \tilde{\mathsf{P}}).$$

Throughout the remainder of this work, the signal process X, the observation process Y and the observation noise process V defined above are interpreted as their natural extensions on the product space Ω without separate notation. For example, the natural extension $X_i^* : \Omega_X \times \Omega_V \times \tilde{\Omega} \to S$ of $X_i : \Omega_X \to S$ is defined for all $\omega = (\omega_X, \omega_V, \tilde{\omega}) \in \Omega_X \times \Omega_V \times \tilde{\Omega}$ as $X_i^*(\omega) = X_i(\omega_X)$. For all $i \in \mathbb{Z}_+$, we let $\mathscr{Y}_i \subset \mathscr{F}$ denote the σ-field generated by the observations $Y_1, Y_2, \ldots, Y_i$ and in particular, $\mathscr{Y}_0 = \{\emptyset, \Omega\}$. Because S and $\mathbb{R}^{d_\mathrm{m}}$ are complete and separable, there exists for all $i \in \mathbb{Z}_+$ *a regular conditional distribution* π_i such that

$$\int \varphi \,\mathrm{d}\pi_i = \mathsf{E}\left[\, \varphi(X_i) \,\middle|\, \mathscr{Y}_i \right],$$

for all bounded and measurable functions $\varphi : S \to \mathbb{R}$. For the proof, see e.g. [64, Theorem 3, page 224 and Corollary, page 228]. The conditional distribution π_i is referred to as *the filtering distribution* and the sequence $\pi \triangleq (\pi_i)_{i \geq 0}$ is referred to as *the filter process* or more shortly *the filter*.

In practice, the filtering problem arises when there is an unknown quantity which is to be estimated based on a set of observations that are obtained through some noisy observation procedure. Classical examples of such applications are for instance the tracking of aircraft using radar observations or the positioning of mobile devices using *the Global Positioning System*. In this case, the unknown quantity is modelled by a stochastic process X and the observations corrupted by a random noise are modelled by a stochastic process Y. Both of these processes are assumed to be of the form defined above. Because of the interpretation that π_i is the conditional distribution of an unobserved random variable X_i given the set of observations $Y_1, Y_2 \dots, Y_i$, the filter π is also known as *the Bayesian filter* and π_i is called *the Bayesian posterior distribution*. Roughly speaking, π_i can be considered as a representation of all the knowledge of X_i given the information provided by the observations $Y_1, Y_2, \dots, Y_i$. It should be noted that in practice, one is typically not interested in π_i itself but some point estimates of X_i or some other statistics. Such quantities can be computed or approximated once the distribution π_i is available and therefore we find ourselves ultimately interested in computing the filtering distribution π_i.

According to the definition, X is Markovian and therefore the filter process π has a recursive representation

$$\pi_i = Q_i(\pi_{i-1}), \qquad \pi_0 = \mathsf{P}_0 \tag{1.2}$$

where the random mapping Q_i will be specified later in Section 2.1. From the practical point of view, the recursive formulation is appealing, because in order to compute π_i, only Y_i and π_{i-1} need to be stored into the memory of the computer instead of the whole observation sequence $Y_1, Y_2, \dots, Y_i$, which would inevitably lead to unbounded memory consumption. In general, however, there is no guarantee that π_{i-1} can be stored in finite memory or that the mapping Q_i can be evaluated. Therefore in practice, Q_i is approximated by Q_i^θ such that the evaluation of Q_i^θ is tractable and the outcome can be stored in a finite memory. Here $\theta \in (0, \infty)$ is a parameter of the approximation such that a larger value of θ implies more accurate approximation but also in practice a higher computational cost of evaluating the approximation. This approximation can be deterministic in the sense that a certain realisation of the observations $Y_1, Y_2, \dots, Y_i$ always produces the same outcome, or it can be random. In order to incorporate this randomness into the analysis, the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathsf{P}})$ was introduced earlier. Similarly, it is observed that the randomness of the signal is represented by the probability space $(\Omega_X, \mathscr{F}_X, \mathsf{P}_X)$ and the randomness of the observation noise is represented by the probability space $(\Omega_V, \mathscr{F}_V, \mathsf{P}_V)$.

Obviously, the approximate mapping Q_i^θ must satisfy some criteria before it can be proposed to be used for approximating the intractable filter recursion, or before it can even be justly said to produce an approximation of the filter. One justification is to show that for all $i > 0$,

$$\lim_{\theta \to \infty} d\Big(Q_i \circ Q_{i-1} \circ \cdots \circ Q_1(\pi_0),\, Q_i^\theta \circ Q_{i-1}^\theta \circ \cdots \circ Q_1^\theta(\pi_0)\Big) = 0, \qquad \mathsf{P}\text{-a.s.} \tag{1.3}$$

where d is a distance defined for probability measures. If d is finite, then according to the dominated convergence theorem, (1.3) also implies the convergence in the mean sense, i.e.

$$\lim_{\theta\to\infty} \mathsf{E}\left[d\left(Q_i \circ Q_{i-1} \circ \cdots \circ Q_1(\pi_0),\, Q_i^\theta \circ Q_{i-1}^\theta \circ \cdots \circ Q_1^\theta(\pi_0)\right)\right] = 0.$$

In this case, the approximation is said to *converge*. From the practical point of view the convergence implies that on a finite time interval, the error of the approximation can be made as small as desired by letting the computational cost of evaluating the approximation increase.

In many cases, the filter will be run for an indefinite time and therefore a stronger form of convergence, namely *uniform convergence*, is desired. By this we mean that

$$\lim_{\theta\to\infty} \sup_{i>0} d\left(Q_i \circ Q_{i-1} \circ \cdots \circ Q_1(\pi_0),\, Q_i^\theta \circ Q_{i-1}^\theta \circ \cdots \circ Q_1^\theta(\pi_0)\right) = 0, \qquad \mathsf{P}\text{-a.s.},$$

and the uniform convergence in the mean sense is defined analogously. The practical interpretation of the uniform convergence is that the error of the approximation with a fixed computational cost is guaranteed to remain below some level and that this level can be made as small as desired by letting the computational cost of evaluating the approximation increase.

It has been observed by several authors that the ability to approximate the filter in a uniformly convergent manner is closely related to the stability properties of the exact filter with respect to its initial conditions. This relation is manifested by results stating that under some conditions a stable filter admits uniformly convergent approximations. Therefore the stability of the filter plays an important role in the analysis of the uniform convergence of various filter approximation methods. The filter is said to be *stable* if for all probability measures $\bar{\pi}_0$ defined on $\mathscr{S}$ one has

$$\lim_{i\to\infty} d\left(Q_i \circ Q_{i-1} \circ \cdots \circ Q_1(\pi_0),\, Q_i \circ Q_{i-1} \circ \cdots \circ Q_1(\bar{\pi}_0)\right) = 0, \qquad \mathsf{P}\text{-a.s.}$$

There is also another motivation for the study of stability. In some applications, it may be the case that the signal is modelled relatively accurately by the transition probabilities K_i but the initial distribution π_0 of the signal is unknown and therefore an erroneous initial distribution $\bar{\pi}_0$ is used instead. In the case of a stable filter, the effect of the incorrect initialisation will eventually disappear and therefore the filter with incorrect initial distribution is guaranteed to perform well in the long run. Considerations regarding the modelling errors are of course not restricted to the choice of initial distribution. In practice, one also makes errors when specifying the transition probabilities K_i, measurement functions h_i, and the measurement noise distributions. Such considerations are, however, left beyond the scope of this thesis.

In this thesis, the focus will be on the stability of the discrete time filter with respect to its initial conditions and on the uniform convergence of certain filter approximation methods.

1.1 Literature review

The theory of the discrete time filter and its numerical approximations is inherently related to probability theory, which is a combination of real analysis and measure theory. Many elementary results and details regarding these branches of mathematics have been excluded from this work but they can be found e.g. in [7, 61, 29, 43, 33] regarding real analysis and measure theory and in [64, 53, 57, 9, 69] regarding general probability theory. Certainly other excellent references exist as well but the ones listed above in particular have been found useful by the author in the course of writing this thesis. Let us then review in more detail the existing literature regarding filter stability and the uniform convergence of filter approximations and point out the references that are most relevant to this work.

Although the computation of π in general is intractable, there are cases for which π can be computed exactly. Perhaps the best-known example of such a situation is the linear-Gaussian case, where X_0 is assumed to be a normally distributed random variable with a known mean and a known covariance, and for all $i > 0$, X_i and Y_i are assumed to be linear functions of X_{i-1} and X_i, respectively, with additive Gaussian noise with a known mean and a known covariance. In this case, π_i, $i > 0$ can be shown to be a normal distribution with a mean and a covariance that can be computed exactly using the well known *Kalman filter* recursion [see e.g. 40, 2, 38, 49, 41]. Because of the tractability of the linear-Gaussian case the first filter approximation methods were based on linearisation such as *the extended Kalman filter (EKF)* [see e.g., 2] or linearisation and the use of Gaussian mixtures [1]. As a result of the increased computational power of computers a novel approach to filter approximation was introduced in [32] by Gordon, Salmond, and Smith. This method was called *the bootstrap filter* as it was essentially an application of *the weighted bootstrap* principle described in [65]. Earlier, the weighted bootstrap principle was referred to as *the sampling/importance resampling (SIR)* algorithm in [62] and therefore the bootstrap filter has also become known as *the SIR filter* [see e.g., 13, 58].

Ever since the introduction of the bootstrap filter, several improvements have been proposed regarding the choice of the sampling or *the importance* distributions [68, 28], *the resampling scheme* [15, 42, 48, 13], or some more innovative improvements such as the addition of *Markov chain Monte Carlo (MCMC)* steps [8], bridging density approach [31] or application of kernel density estimation methods [52, 37]. More comprehensive reviews on the existing methods can be found in [59, 27, 3]. Because the methods mentioned above are all based on the simulation of random variables, the methods have become commonly known as *sequential Monte Carlo (SMC)* methods. *The Monte Carlo method* itself is a random variable simulation based method for approximating integrals and dates back to 1940's. The theory of the Monte Carlo method is extensive and more details can be found e.g. in [34, 60, 63, 51, 30].

The application of the Monte Carlo method, the weighted bootstrap, or the SIR algorithm represents a statistical approach to the filter approximation problem. Another approach, based on a more stochastic point of view was developed simultaneously in [19] which is also one of the earliest contributions regarding the convergence of SMC methods. In [19], an algorithm called *the interacting particle filter (IPF)* was introduced and it was shown to be convergent. The IPF algorithm was essentially the bootstrap filter, but the terminology originates from the interpretation that the random samples are regarded as *particles* that evolve in the state space according to some stochastic dynamics and the term *interacting* refers to the fact that the particles are not independent of each other. Later, the convergence results were extended to cover more general classes of filter approximation algorithms by allowing various branching schemes to be employed by the approximating algorithm [see e.g., 20, 18]. In [16, 17], the convergence results are further extended to more general and practical class of SIR filters. Although historically the term *particle* has been used when referring to a realisation of a random variable taking values in the state space, in principle, particles can be endowed with a much more elaborate structure. For instance, they can be defined as normal distributions in the state space. In [15], this possibility for generalisation has been retained in the convergence analysis, and not only sufficient but also necessary conditions for the convergence of *a particle filter* are given without making any assumptions on the structure of the particles.

Compared to the results on the convergence of filter approximations, the results on the uniform convergence are naturally more scarce and restricted. As mentioned in the previous section, there is a close relation between the ability to approximate the filter in a uniformly convergent manner and the stability of the filter with respect to its initial conditions. One of the earliest contributions regarding the uniform convergence of particle filter approximations is [20] which introduced a finite memory length *Monte Carlo particle filter* and proved that the approximation is uniformly convergent if the filter itself is stable. Later the connection between the stability and the uniform convergence has been studied e.g. in [22, 47, 56].

In the case of linear filters, i.e. the linear-Gaussian model in discrete time, the analysis of the filter stability is reduced to considering the stability of the difference equations describing the behaviour of the mean and the covariance process of the filter. For these processes it is known that the stability is equivalent to the complete stabilisability and detectability of the signal and observation model. See e.g., [2, pages 76-82] for the discrete time case and [46, Theorem 4.11] and [54] for the continuous time case. For the nonlinear filter, the mean and the covariance are not sufficient for characterising the filtering distribution entirely and therefore similar approaches cannot be applied.

In [66], the stability of the filter for linear-Gaussian model with non-Gaussian initial distribution has been considered in the sense that the error between the mean of the exact filtering distribution and the estimate given by the Kalman recurrence vanishes as time goes to infinity. Similar result for the continuous time case is given in [54]. Also the stability of a more general class of nonlinear continuous time filters is studied in [54] by Ocone and Pardoux and it is shown that if the signal process satisfies certain *ergodicity* assumptions, then the filter is stable. The theory in [54] is developed on the results by Stettner [67] and Kunita [44] who have studied the possibility to lift the ergodicity properties of the signal to the filter process. However, a gap in the proof of one of the key results in [44] has been discovered recently, as pointed out in [10] and some of the results in [54] and [67] are affected by the gap as well. This problem remains unsolved, but it has been pointed out in [10] that a solution to the problem would imply a number of desired properties for the filter process, including stability.

The problems in the analysis of the filter stability are mainly due to the nonlinearity of Q_i which follows from the normalisation incorporated in Q_i (see Section 2.1 for details). A distance defined for finite measures known as *the Hilbert metric* is scaling invariant and thus the problems arising from the normalisation can be avoided with the Hilbert metric. Moreover, it is known that the Hilbert metric can be used for bounding *the total variation distance* between two measures from above and therefore it provides a powerful tool for the stability analysis [see e.g., 47]. Atar and Zeitouni [5] use the Hilbert metric to prove the filter stability under various conditions. Most notably, the signal is assumed to be ergodic and in addition *mixing* or to take values in a compact state space. It should be noted that also the mixing condition can usually be shown to hold only for signals taking values in a compact state space. More details on the mixing condition can be found e.g. in [47] where the mixing condition on the signal kernel is replaced by the mixing condition on the unnormalised filter process kernel which is a slightly weaker but nevertheless rather strong condition. Also the method in [47] uses the Hilbert metric. A downside of the Hilbert metric is that it is typically applicable only for compact state spaces. For this reason the references above appear to require either a compact state space explicitly or some mixing conditions that usually hold only for compact state spaces.

Another approach to study the filter stability without using the Hilbert metric was developed by Del Moral, Guionnet and Miclo in [22] and [23, Ch. 2] (see also [21, Ch. 4]). This method was based on the use of *the Dobrushin ergodic coefficient* [25, 26] and by using this approach the stability of filter with exponential convergence rate was proved in [22, 23, 21]. Nevertheless, the signal was assumed to satisfy some mixing type conditions. It should be pointed out that the method of the Dobrushin ergodic coefficient described in [22] forms also the basis for the analysis in this work.

In addition to pioneering the use of the Hilbert metric, it was also shown in [5] that in the case of a bounded one dimensional state space the convergence rate can be increased without bound by letting the observation noise go to zero. This is one of the first results explicitly stating a relation between the filter stability and the accuracy of the observations. This relation was again pointed out in [4], where the filtering of a one dimensional ergodic diffusion in a noncompact state space was considered. It was shown that the filter is stable, provided that the observation noise is sufficiently small. It was also shown that the convergence rate can be increased without a bound by letting the observation noise go to zero.

All the stability results mentioned above assumed that the signal itself is well-behaved, e.g. ergodic or mixing. One of the first contributions regarding the nonergodic case is [11] where the Hilbert metric was used for proving the exponential stability of the filter in the case of possibly nonergodic signal. Because of the use of the Hilbert metric, the observation noise was assumed to be bounded which ensures a compact support for the filtering distribution and therefore the Hilbert metric is applicable. Also the boundedness of the observation noise can be interpreted as a requirement for sufficiently accurate observations. In [12], a method which had earlier appeared in [5] and [4] was used for relaxing the assumption about the boundedness of the observation noise. This method was based on bounding the total variation distance between measures by using *the exterior product* of the unnormalised densities. The filter was shown to be exponentially stable provided that the observation noise is sufficiently small. More recently, the stability in the case of nonergodic signal has been studied also in [56] where the method of the Dobrushin ergodic coefficient is further developed and it is shown that the stability of the filter for nonergodic signal does not necessarily require the observation noise to be small as long as it is sufficiently light tailed compared to the signal noise. In this work, we further extend the results reported in [56].

Of all the references mentioned above, the uniform convergence of filter approximations is addressed only in [20], [23], [22], [21, Ch. 7], [47] and [56]. In [20], [23], [22], [21] and [47] the stability of the filter with some additional assumptions imposed on the approximating algorithm has been shown to imply the uniform convergence. In [56], the stability and the uniform convergence have been studied separately but it is shown that similar conditions on the filter framework appear to imply both the stability and the uniform convergence. Moreover, the uniform convergence of particle filter approximations has also been proved in [45] in the case of a mixing signal.

1.2 Contributions and organisation

This work is based on the method of the Dobrushin ergodic coefficient described in [22] and it can be regarded as an extension and a refinement of the results reported in [56]. The main contributions regarding the filter stability are due to the fact that to a large extent the analysis of the stability in this work is done in the almost sure sense and therefore it is substantially different from the approach described in [56]. Several benefits are obtained from this approach:

- The extension of the stability theorem to more general signal and observation noise distributions is straightforward because almost sure bounds for the signal and observation noise terms are easily obtained in the almost sure sense (see Proposition 2.10).
- The analysis provides convergence rates directly in the almost sure sense which implies the convergence in the mean sense by the dominated convergence theorem.
- The stability analysis is similarly based on the filter approximation by truncation as in [56] but in addition, almost sure nonuniform bounds for the error of the truncated filter approximation are obtained. This implies the almost sure convergence of the truncated filter approximation. Also rates for the convergence are obtained.
- The stability result in [56] holds only for filters whose initial distribution is comparable to the exact initial distribution in the sense of the Hilbert metric. This rather restrictive assumption is avoided by the approach used here.

Also in the analysis of the uniform convergence, there are many similarities between this work and [56]. The main contributions regarding the uniform convergence are the following:

- The uniform convergence of a certain rejection method based SMC algorithm has been proved in [56]. The convergence is proved with respect to the number of accepted samples which does not represent the computational cost of the approximation. In fact, the possibility that the average number of the rejected samples, and consequently the computational cost, increases over time has not been addressed. Therefore to some extent the uniform convergence with respect to the computational cost has not been proved. In this work, the uniform convergence is proved for a general class of filter approximation algorithms, including a SIR filter type algorithm. For this algorithm, the computational cost is determined by the sample size and therefore the uniform convergence is obtained with respect to the computational cost.

- The conditions for the approximating algorithm are given in a general form. Because of this generality, the uniform convergence can be proved straightforwardly for several different filter approximation methods. Examples of such approximation algorithms are the above mentioned SIR filter type algorithms with different resampling schemes or certain rejection method based approximations. Note that in the case of the rejection method, the uniform convergence can only be obtained with respect to the expected computational cost.

The remainder of this work is organised as follows.

Chapter 2: This chapter is focused on the filter stability. Moreover, it includes some background and preliminaries that are needed also in Chapter 3. Section 2.1 defines and proves the well known recursive formulation of the discrete time filter which provides the basis for the analysis of both the stability and the uniform convergence. Section 2.2 defines the elementary concepts of random probability measures and various forms of their convergence. These definitions are needed in order to specify rigorously what is meant by the stability and the uniform convergence. Section 2.3 introduces the method of approximating the filter by truncating the support of the filtering distribution. Moreover, the section includes a discussion about the principles of how the truncated approximation can be applied to proving the stability. In Section 2.4, the filter framework under consideration is specified in detail by stating a list of assumptions about the signal process and the observation process. The chapter is concluded in Section 2.5 which states the main stability result and a corollary which establishes convergence rates for the stability.

Chapter 3: This chapter is focused on the uniform convergence of filter approximations. Section 3.1 includes some preliminary results accompanied by a theorem which establishes easily verifiable sufficient conditions for the uniform convergence of the truncated filter with respect to the truncation radius. Section 3.2 specifies the set of approximating algorithms under consideration by imposing some general conditions on the approximating algorithms. Also, the section states the main result on the uniform convergence as well as a practical corollary which establishes the uniform convergence of certain point estimates. In Section 3.3 the set of uniformly convergent filter approximations is exemplified by introducing a feasible SIR filter type algorithm which is shown to satisfy the conditions for uniform convergence. The chapter is concluded in Section 3.4 by illustrating the uniform convergence results by some computer simulations.

Chapter 4: This chapter concludes the thesis with some remarks and conclusions regarding the results. A discussion about the stability and the uniform convergence results are provided in Section 4.1 and Section 4.2, respectively. Section 4.3 points out some topics for future research.

1.3 Notations

Although this work follows the typical notational conventions used in the related literature, some of the notations may not be entirely self-explanatory, in particular, to a reader who is unfamiliar with the topic. Therefore this section explains some of the general notations used throughout the remainder of this work.

The symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{Q}$ are used for denoting the sets of real numbers, nonnegative real numbers, natural numbers, nonnegative integers and rational numbers, respectively. The space $\mathbb{R}^n$ is assumed to be endowed with the topology induced by the Euclidean norm which is denoted by $\|\cdot\|$. The measurability of functions taking values in some subset of $\mathbb{R}^n$ is always considered in terms of the Borel field in the given subset of $\mathbb{R}^n$. The Borel field in $\mathbb{R}^n$ is denoted by $\mathscr{B}(\mathbb{R}^n)$.

The set of continuous and bounded functions $\varphi : S \to \mathbb{R}$ is denoted by $C_\mathrm{b}(S)$ and the subset of $C_\mathrm{b}(S)$ consisting of the compactly supported functions is denoted by $C_\mathrm{c}(S)$. These spaces are endowed with the topology induced by the supremum norm

$$\left\|\varphi\right\|_\infty \triangleq \sup_{x\in S}\left|\varphi(x)\right|.$$

The set of probability measures and finite measures on an arbitrary σ-field $\mathscr{S}$ are denoted by $M_\mathrm{P}(\mathscr{S})$ and $M_\mathrm{F}(\mathscr{S})$, respectively. The set of bounded and measurable mappings $\varphi : S \to \mathbb{R}$ is denoted by $B(\mathscr{S})$. For all $\varphi \in B(\mathscr{S})$ and for all measures μ on $\mathscr{S}$ we define

$$\mu(\varphi) \triangleq \int \varphi \, \mathrm{d}\mu$$

When confusion will not arise, the parentheses can be omitted. If, in particular, $\varphi = 1_A$ where 1_A denotes the indicator function of a set $A \in \mathscr{S}$, then the shorthand notation $\mu(A) \triangleq \mu(1_A)$ is used. The Lebesgue measure on $\mathscr{B}(\mathbb{R}^n)$ is denoted by λ_n and for all λ_n-integrable functions φ, the conventional notation $\int \varphi(x)\,\mathrm{d}x = \int \varphi \,\mathrm{d}\lambda_n$ is used.

Suppose that $K : S \times \mathscr{S} \to [0, 1]$ is a transition probability. Then, for all $\varphi \in B(\mathscr{S})$ we define the function $K(\varphi): S \to \mathbb{R}$ in $B(\mathscr{S})$ such that

$$K(\varphi)(x) = \int \varphi(y) K(x, \mathrm{d}y),$$

for all $x \in S$. Moreover, for all $\mu \in M_{\mathrm{P}}(\mathscr{S})$, we define a probability measure $\mu K \in M_{\mathrm{P}}(\mathscr{S})$ such that

$$\mu K(\varphi) = \iint \varphi(y) K(x, \mathrm{d}y)\, \mu(\mathrm{d}x), \tag{1.4}$$

for all $\varphi \in B(\mathscr{S})$. Accordingly, for multiple transition probabilities $K_1, K_2, \ldots, K_i$, the measure $\mu K_1 K_2 \cdots K_i$ satisfies

$$\mu K_1 K_2 \cdots K_i(\varphi) = \iint \cdots \int \varphi(x_i) K_i(x_{i-1}, \mathrm{d}x_i) \cdots K_2(x_1, \mathrm{d}x_2) K_1(x_0, \mathrm{d}x_1) \mu(\mathrm{d}x_0),$$

for all $\varphi \in B(\mathscr{S})$.

Let $\nu \in M_{\mathrm{P}}(\mathscr{S})$ be arbitrary. For all nonnegative $\psi \in B(\mathscr{S})$ and for all $\mu \in M_{\mathrm{F}}(\mathscr{S})$, we define a probability measure $\psi \cdot \mu \in M_{\mathrm{P}}(\mathscr{S})$ such that

$$(\psi \cdot \mu)(\varphi) = \begin{cases} \dfrac{\mu(\psi\varphi)}{\mu(\psi)} & \text{if } \mu(\psi) > 0 \\ \nu(\varphi) & \text{otherwise }, \end{cases}$$

for all $\varphi \in B(\mathscr{S})$.

For any set A of random variables defined in Ω, the notation $\sigma(A)$ is used for denoting the σ-field generated by the random variables in A.

For all $\varphi, \psi \in B(\mathscr{B}(\mathbb{R}^d))$, the convolution of φ and ψ is denoted by $\varphi * \psi$, i.e. for all $x \in \mathbb{R}^d$,

$$(\varphi * \psi)(x) = \int \varphi(x - y) \psi(y) \mathrm{d}y.$$

Also, we define $\exp(x) \triangleq \mathrm{e}^x$ for all $x \in \mathbb{R}$.

Chapter 2

Stability

This chapter is focused on the stability of the discrete time filter with respect to its initial conditions but it also contains general background and preliminaries that are needed in Chapter 3. The proof of the stability is based on approximating the exact filter recursion by truncating the support of the filtering distribution such that the resulting support is a compact subset of the state space. Because of the compactness of the support, it is not difficult to show that the resulting approximation is stable. In fact, it follows that the approximation is sufficiently stable as well as a sufficiently good approximation of the exact filter recursion in the sense that it can be parameterised such that

- the distance between approximate filters with different initial distributions converges to zero as time increases
- the approximation error converges to zero for all initial distributions as time increases.

This observation then implies the stability for the exact filter by a simple application of the triangle inequality.

This chapter is organised as follows. Section 2.1 defines and proves the well known recursive formulation of the discrete time filter which provides the basis for the analysis of both the stability and the uniform convergence. Section 2.2 defines the elementary concepts of random probability measures and various forms of their convergence. These definitions are needed in order to specify rigorously what is meant by the stability and the uniform convergence. Section 2.3 introduces the method of approximating the filter by truncating the support of the filtering distribution. Moreover, the section includes a discussion about the principles of how the truncated approximation can be applied to proving the stability. It should be pointed out that the majority of the material covered by sections 2.1, 2.2, and 2.3

consist of known results and preliminaries that are also needed in Chapter 3. In Section 2.4, the filter framework under consideration is specified in detail by stating a list of assumptions about the signal process and the observation process. These assumptions are then retained throughout the remainder of this work. Also a number of important intermediate results are proved. In Section 2.5, the main result regarding the filter stability is proved accompanied by a corollary which establishes convergence rates for the stability.

2.1 Exact filter recursion

Throughout the remainder of this work it is assumed that the random variable V_i, $i > 0$ has a continuous positive density ρ_{V_i} with respect to λ_{d_m}. In this case, we define

$$g_{i,y}(x) \triangleq \rho_{V_i}(y - h_i(x)),$$

for all $i > 0$, $y \in \mathbb{R}^{d_\mathrm{m}}$ and $x \in \mathbb{R}^{d_\mathrm{s}}$. Moreover, we define the shorthand notation

$$g_i(x) \triangleq g_{i,Y_i}(x),$$

for all $i > 0$ and $x \in \mathbb{R}^{d_\mathrm{s}}$. In the literature, the function g_i is commonly referred to as *the likelihood function*. For convenience, we also define for all $i > 0$, $X_{0:i-1} \triangleq (X_0, X_1, \ldots, X_{i-1})$ and $Y_{1:i} \triangleq (Y_1, Y_2, \ldots, Y_i)$. In this case, it follows from the definition of X, Y and V that

$$\mathsf{P}(Y_i \in A_i \mid X_{0:i}, Y_{1:i-1}) = \mathsf{P}(Y_i \in A_i \mid X_i) = \int_{A_i} g_{i,y}(X_i) \mathrm{d}y. \tag{2.1}$$

We then have the following fundamental and well known result which yields a recursive formulation of the discrete time filter. The proof of the theorem has been included for completeness and it can also be found e.g. in [15].

Theorem 2.1. *For all $i > 0$,*

$$\pi_i = g_i \cdot \pi_{i-1} K_i, \qquad \mathsf{P}\text{-a.s.}$$

Proof. Let us define a signed measure $Q : \mathscr{B}((\mathbb{R}^{d_\mathrm{s}})^i) \to \mathbb{R}$ such that for all $A \in \mathscr{B}((\mathbb{R}^{d_\mathrm{s}})^i)$

$$Q(A) = \int_{\{Y_{1:i} \in A\}} \varphi(X_i) \mathrm{d}\mathsf{P},$$

where $\{Y_{1:i} \in A\} \triangleq \{\omega \in \Omega \mid (Y_1(\omega), Y_2(\omega), \ldots, Y_i(\omega)) \in A\}$. Suppose that $A_{1:i} = \prod_{j=1}^{i} A_j$, where $A_j \in \mathscr{B}(\mathbb{R}^{d_m})$, $1 \le j \le i$. In this case,

$$Q(A_{1:i}) = \int \varphi(X_i)\mathsf{P}(Y_{1:i} \in A_{1:i} \mid X_{0:i})\mathrm{d}\mathsf{P}.$$

By defining $\{Y_i \in A_i\} \triangleq \{\omega \in \Omega \mid Y_i(\omega) \in A_i\}$, we can write

$$\begin{aligned}\mathsf{P}(Y_{1:i} \in A_{1:i} \mid X_{0:i}) &= \mathsf{E}\left[\mathsf{E}\left[\prod_{j=1}^{i} 1_{\{Y_j \in A_j\}} \middle| X_{0:i}, Y_{1:i-1} \right] \middle| X_{0:i} \right] \\ &= \mathsf{E}\left[\prod_{j=1}^{i-1} 1_{\{Y_j \in A_j\}} \middle| X_{0:i} \right] \mathsf{P}(Y_i \in A_i \mid X_i), \end{aligned} \tag{2.2}$$

from which it follows by repeating the same reasoning for the remaining conditional expectation in (2.2) and by using (2.1) that

$$\mathsf{P}(Y_{1:i} \in A_{1:i} \mid X_{0:i}) = \prod_{j=1}^{i} \mathsf{P}(Y_j \in A_j \mid X_i) = \int \prod_{j=1}^{i} g_{i,y_i}(X_i)\mathrm{d}y_{1:i},$$

where the integral is a shorthand notation for an i times iterated integral with respect to the Lebesgue measure. By using the definition of X, Fubini's theorem, and Carathéodory's extension theorem, one has for all $A \in \mathscr{B}((\mathbb{R}^{d_s})^i)$

$$Q(A) = \int_A \left[\int \varphi(x_i) \prod_{j=1}^{i} g_{j,y_j}(x_j) K_i(x_{i-1}, \mathrm{d}x_i) K_{i-1}(x_{i-2}, \mathrm{d}x_{i-1}) \cdots \pi_0(\mathrm{d}x_0) \right] \mathrm{d}y_{1:i}.$$

Similarly, the above reasoning in the case $\varphi \equiv 1$ yields for all $A \in \mathscr{B}((\mathbb{R}^{d_s})^i)$

$$\mathsf{P}_{Y_{1:i}}(A) = \int_A \left[\int \prod_{j=1}^{i} g_{j,y_j}(x_j) K_i(x_{i-1}, \mathrm{d}x_i) K_{i-1}(x_{i-2}, \mathrm{d}x_{i-1}) \cdots \pi_0(\mathrm{d}x_0) \right] \mathrm{d}y_{1:i},$$

where $\mathsf{P}_{Y_{1:i}}$ denotes the distribution of $Y_{1:i}$. Then, [see e.g., 64, pages 219 and 230]

$$\pi_i \varphi = \frac{\mathrm{d}Q}{\mathrm{d}\mathsf{P}_{Y_{1:i}}}(Y_{1:i}) = \frac{\int \varphi(x_i) \prod_{j=1}^{i} g_{j,Y_j}(x_j) K_i(x_{i-1}, \mathrm{d}x_i) K_{i-1}(x_{i-2}, \mathrm{d}x_{i-1}) \cdots \pi_0(\mathrm{d}x_0)}{\int \prod_{j=1}^{i} g_{j,Y_j}(x_j) K_i(x_{i-1}, \mathrm{d}x_i) K_{i-1}(x_{i-2}, \mathrm{d}x_{i-1}) \cdots \pi_0(\mathrm{d}x_0)},$$

from which the claim follows by induction. □

For all $i > 0$ we define Q_i to be the mapping $\pi_{i-1} \mapsto \pi_i = g_i \cdot \pi_{i-1} K_i$, which yields the recursive formulation of π given in (1.2).

It is also observed that by the independence of X and V and the Markovian property of

X, one has for all $\varphi \in B(\mathscr{S})$ and $i > 0$

$$\begin{aligned} \mathsf{E}\big[\varphi(X_i) \mid Y_{1:i-1}\big] &= \mathsf{E}\big[\mathsf{E}\big[\varphi(X_i) \mid X_{0:i-1}, V_{1:i-1}\big] \mid Y_{1:i-1}\big] \\ &= \mathsf{E}\big[\mathsf{E}\big[\varphi(X_i) \mid X_{i-1}\big] \mid Y_{1:i-1}\big] = \pi_{i-1} K_i \varphi. \end{aligned} \tag{2.3}$$

Therefore it follows from (2.1) and Fubini's theorem that

$$\begin{aligned} \mathsf{P}(\{Y_i \in A\} \cap \{Y_{1:i-1} \in B\}) &= \int_{\{Y_{1:i-1} \in B\}} \mathsf{E}\big[\mathsf{P}(Y_i \in A_i \mid X_i, Y_{1:i-1}) \mid Y_{1:i-1}\big] \, \mathrm{d}\mathsf{P} \\ &= \int_{\{Y_{1:i-1} \in B\}} \int_A \int g_{i,y}(x_i) \pi_{i-1} K_i(\mathrm{d}x_i) \mathrm{d}y \, \mathrm{d}\mathsf{P}, \end{aligned}$$

from which we conclude that

$$\mathsf{P}(Y_i \in A \mid \mathscr{Y}_{i-1}) = \int_A \int g_{i,y}(x_i) \pi_{i-1} K_i(\mathrm{d}x_i) \, \mathrm{d}y,$$

and consequently

$$\mathsf{E}\big[\varphi(Y_i) \mid \mathscr{Y}_{i-1}\big] = \int \varphi(y) \int g_{i,y}(x_i) \pi_{i-1} K_t(\mathrm{d}x_i) \, \mathrm{d}y. \tag{2.4}$$

This equality will be needed later in Section 3.1.

2.2 Convergence of random probability measures

In order to define the uniform convergence of filter approximations rigorously we need to specify what is meant by the convergence of probability measures on $\mathscr{S}$. Therefore, a topological structure in $M_\mathrm{P}(\mathscr{S})$ is required. Once the topological structure is fixed, we have access to the associated Borel field which also enables a rigorous definition of *random probability measures*. Moreover, if the topology in $M_\mathrm{P}(\mathscr{S})$ is metrisable, then also the stability of the filter is well defined.

2.2.1 Weak convergence

The space $M_\mathrm{P}(\mathscr{S})$ is endowed with *the weak topology* which is the smallest topology that makes the mappings $\mu \in M_\mathrm{P}(\mathscr{S}) \mapsto \mu(\varphi) \in \mathbb{R}$ continuous for all $\varphi \in C_\mathrm{b}(S)$. Equivalently, the weak topology can be defined as a topology whose base consists of the sets

$$V_{\varphi_1,\varphi_2,\ldots,\varphi_n;\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}(\mu) \triangleq \Big\{\nu \in M_\mathrm{P}(\mathscr{S}) \;\Big|\; \big|\nu\varphi_i - \mu\varphi_i\big| < \varepsilon_i, 1 \le i \le n\Big\}$$

where $\mu \in M_{\mathrm{P}}(\mathscr{S})$, $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a finite subset of $C_{\mathrm{b}}(S)$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are positive real numbers [see e.g., 57, 7]. Naturally, *the weak convergence* of a sequence in $M_{\mathrm{P}}(\mathscr{S})$ is then defined to mean the convergence in the weak topology. Another equivalent characterisation of the weak convergence is obtained by observing that a sequence $(\mu_i)_{i\geq 0}$ in $M_{\mathrm{P}}(\mathscr{S})$ converges weakly to $\mu \in M_{\mathrm{P}}(\mathscr{S})$ if and only if

$$\lim_{i\to\infty} \mu_i \varphi = \mu\varphi, \quad \forall \varphi \in C_{\mathrm{b}}(S). \tag{2.5}$$

In fact, (2.5) is used as the definition of weak convergence in several references instead of the topological definition given above [see e.g, 7, 9].

Because S is locally compact and second countable, it is *Polish* [see e.g., 7, Theorem 7.6.1]. This implies that $C_{\mathrm{c}}(S)$ is separable and therefore there exists a countable and dense subset $\hat{\Phi} = \{\hat{\varphi}_1, \hat{\varphi}_2, \ldots\} \subset C_{\mathrm{c}}(S)$ [see e.g., 7, Theorem 7.6.3]. In this case, one can define a function $d_{\mathrm{w}} : M_{\mathrm{P}}(S) \times M_{\mathrm{P}}(S) \to [0, 1]$ as

$$d_{\mathrm{w}}(\mu, \nu) \triangleq \frac{1}{2} \sum_{i=1}^{\infty} \frac{\left|\mu\hat{\varphi}_i - \nu\hat{\varphi}_i\right|}{2^i \left\|\hat{\varphi}_i\right\|_\infty}. \tag{2.6}$$

By the denseness of $\hat{\Phi}$ and Riesz's representation theorem [see e.g., 7, Theorem 7.5.4], it follows that d_{w} is a metric in $M_{\mathrm{P}}(\mathscr{S})$. Note that the choice of the set $\hat{\Phi}$ is not unique and therefore also the definition of d_{w} is not unique. Moreover, different metrics in $M_{\mathrm{P}}(\mathscr{S})$ are obtained by using different permutations of the functions in $\hat{\Phi}$. However, it can be shown that all the resulting metrics are topologically equivalent and that the induced topology is precisely the weak topology. Thus in the case of Polish state space S, (2.5) is equivalent to

$$\lim_{i\to\infty} d_{\mathrm{w}}(\mu_i, \mu) = 0. \tag{2.7}$$

2.2.2 Random probability measures and transition probabilities

The topological structure of $M_{\mathrm{P}}(\mathscr{S})$ defined above also enables us to introduce the associated Borel field in $M_{\mathrm{P}}(\mathscr{S})$ which is denoted by $\mathscr{M}_{\mathrm{P}}(\mathscr{S})$. In this case, we say that a mapping $\mu : \Omega \to M_{\mathrm{P}}(\mathscr{S})$ is a random probability measure if it is $\mathscr{F}/\mathscr{M}_{\mathrm{P}}(\mathscr{S})$-measurable. The following well known result gives a convenient formulation for the measurability of random probability measures.

Theorem 2.2. *Suppose that $\mathscr{G} \subset \mathscr{F}$. The mapping $\mu : \Omega \to M_{\mathrm{P}}(\mathscr{S})$ is $\mathscr{G}/\mathscr{M}_{\mathrm{P}}(\mathscr{S})$-measurable if and only if $\mu\varphi : \Omega \to \mathbb{R}$ is $\mathscr{G}/\mathscr{B}(\mathbb{R})$-measurable for all $\varphi \in C_{\mathrm{b}}(S)$.*

Proof. Suppose that μ is $\mathscr{G}/\mathscr{M}_{\mathrm{P}}(\mathscr{S})$-measurable. Because $\mu \mapsto \mu\varphi$, as a continuous function, is $\mathscr{M}_{\mathrm{P}}(\mathscr{S})/\mathscr{B}(\mathbb{R})$-measurable, the composition $\omega \mapsto \mu(\omega) \mapsto (\mu(\omega))(\varphi)$ must be $\mathscr{G}/\mathscr{B}(\mathbb{R})$-measurable.

To prove the converse implication, suppose that $(\mu(\cdot))(\varphi)$ is $\mathscr{G}/\mathscr{B}(\mathbb{R})$-measurable for all $\varphi \in C_{\mathrm{b}}(S)$ and let $\mathscr{M}$ be the smallest σ-field in $M_{\mathrm{P}}(\mathscr{S})$ making $\mu \mapsto \mu\varphi$ $\mathscr{M}/\mathscr{B}(\mathbb{R})$-measurable for all $\varphi \in C_{\mathrm{b}}(S)$. Then μ is $\mathscr{G}/\mathscr{M}$-measurable [see e.g, 7, Theorem 1.7.4] and it suffices to show that $\mathscr{M}_{\mathrm{P}}(\mathscr{S}) \subset \mathscr{M}$. It is an elementary exercise to show that the collection $\mathscr{U}$ of the sets

$$\left\{ \mu \in M_{\mathrm{P}}(\mathscr{S}) \;\middle|\; \frac{1}{2}\sum_{j=1}^{i} \frac{\left|\mu\hat{\varphi}_j - q_j\right|}{2^j \left\|\hat{\varphi}_j\right\|_\infty} < r \right\},$$

where $i \in \mathbb{N}$, $r \in \mathbb{Q}$, $q_j \in \mathbb{Q}$, is a countable base for the weak topology. Thus for all $i \in \mathbb{N}$ and $(q_1, \ldots, q_i) \in \mathbb{Q}^i$, the mapping

$$\mu \mapsto \frac{1}{2}\sum_{j=1}^{i} \frac{\left|\mu\hat{\varphi}_j - q_j\right|}{2^j \left\|\hat{\varphi}_j\right\|_\infty}$$

is $\mathscr{M}/\mathscr{B}(\mathbb{R})$-measurable and hence $\mathscr{U} \subset \mathscr{M}$. Therefore $\mathscr{M}$ contains the weak topology and because $\mathscr{M}_{\mathrm{P}}(S)$ is the smallest such σ-field, one must have $\mathscr{M}_{\mathrm{P}}(S) \subset \mathscr{M}$. □

According to the definition of π, it is now obvious that for all $\varphi \in C_{\mathrm{b}}(S)$ the mapping $\pi_i\varphi : \Omega \to \mathbb{R}$ is $\mathscr{Y}_i/\mathscr{B}(\mathbb{R})$-measurable and hence, according to Theorem 2.2, $\pi_i : \Omega \to M_{\mathrm{P}}(S)$ is $\mathscr{Y}_i/\mathscr{M}_{\mathrm{P}}(S)$-measurable random probability measure. Therefore π can be justly called a probability measure valued $\mathscr{Y}_i$-adapted stochastic process.

Because S is separable and complete, it can be shown that if $\mu : \Omega \to M_{\mathrm{P}}(\mathscr{S})$ is $\mathscr{G}/\mathscr{M}_{\mathrm{P}}(\mathscr{S})$-measurable, then for all $A \in \mathscr{S}$, $(\mu(\cdot))(A)$ is $\mathscr{G}/\mathscr{B}(\mathbb{R})$-measurable [see e.g., 39, Lemma 1.4]. Therefore the mapping $\tilde{\mu} : \Omega \times \mathscr{S} \to [0,1]$ defined for all $\omega \in \Omega$ and $A \in \mathscr{S}$ as $\tilde{\mu}(\omega, A) \triangleq (\mu(\omega))(A)$ is a transition probability. In a similar fashion we can define *random transition probabilities* by saying that $K : \Omega \times S \times \mathscr{S} \to [0,1]$ is a $\mathscr{G}$-measurable random transition probability if for all $(\omega, x) \in \Omega \times S$, $K(\omega, x, \cdot) \in M_{\mathrm{P}}(\mathscr{S})$ and for all $A \in \mathscr{S}$, $K(\cdot, \cdot, A)$ is $\mathscr{G} \otimes \mathscr{S}$-measurable. In this case, $K(\omega, \cdot, \cdot)$ is a transition probability for all $\omega \in \Omega$, and $K(\cdot, x, A)$ as well as $K(\varphi)(\cdot, x) \triangleq \int \varphi(y) K(\cdot, x, \mathrm{d}y)$ are $\mathscr{G}$-measurable random variables. More details can be found e.g. in [53, Proposition III.2.1]. Throughout the remainder of this work, the dependency on ω will not explicitly appear in the notations of random probability measures and transition probabilities. In other words, we will write $\mu(A)$ instead of $(\mu(\cdot))(A)$, $\mu(\varphi)$ instead of $(\mu(\cdot))(\varphi)$, $K(x, A)$ instead of $K(\cdot, x, A)$, and $K(\varphi)(x)$ instead of $K(\varphi)(\cdot, x)$.

2.2.3 Convergence in total variation

A stronger form of convergence is obtained by using the metric induced by the norm $\|\cdot\|_{\mathrm{TV}}$ which is defined for signed measures as

$$\|\mu\|_{\mathrm{TV}} \triangleq \frac{1}{2} \sup_{\|\varphi\|_\infty \leq 1} |\mu(\varphi)|. \tag{2.8}$$

To avoid confusion, it should be noted that $\|\cdot\|_{\mathrm{TV}}$ is half the total variation norm defined e.g. in [50, page 315]. For convenience, the multiplication by $1/2$ is used here because it scales the induced metric to the interval $[0,1]$. Therefore in this work, following the example set by several other authors, $\|\cdot\|_{\mathrm{TV}}$ will be called the total variation norm and the induced metric will be called the total variation distance.

A sequence $(\mu_i)_{i\geq 0}$ in $M_{\mathrm{P}}(\mathscr{S})$ is said to converge to $\mu \in M_{\mathrm{P}}(\mathscr{S})$ in total variation, if

$$\lim_{i\to\infty} \|\mu_i - \mu\|_{\mathrm{TV}} = 0. \tag{2.9}$$

According to (2.6) and (2.8) one clearly has

$$d_{\mathrm{w}}(\mu,\nu) \leq \|\mu - \nu\|_{\mathrm{TV}},$$

and thus the convergence in total variation implies weak convergence. It should also be pointed out that in the case of a locally compact and second countable state space S, the total variation can be shown to satisfy

$$\|\mu\|_{\mathrm{TV}} = \frac{1}{2} \sup_{\varphi\in\hat{\Phi}} \frac{|\mu(\varphi)|}{\|\varphi\|_\infty},$$

and thus for all $\mathscr{G}/\mathscr{M}_{\mathrm{P}}(\mathscr{S})$-measurable random probability measures μ and ν the total variation distance $\|\mu - \nu\|_{\mathrm{TV}}$ is clearly a $\mathscr{G}/\mathscr{B}(\mathbb{R})$-measurable random variable.

The following lemma summarises some elementary properties of the total variation distance. These properties in particular are important for the remainder of this work. Parts of the proof can also be found in [55, Lemma 2.8].

Lemma 2.3. *Suppose that* μ, $\nu \in M_{\mathrm{P}}(\mathscr{S})$, $D \in \mathscr{S}$ *and* $\psi : S \to [0,\infty)$ *such that* $0 < \mu(\psi), \nu(\psi) < \infty$. *Then*

(i) $\|\mu - 1_D \cdot \mu\|_{\mathrm{TV}} = \mu(\complement D)$;

(ii) $\|\psi\cdot\mu - \psi\cdot\nu\|_{\mathrm{TV}} \leq \dfrac{\|\psi\|_\infty}{\mu(\psi)} \|\mu - \nu\|_{\mathrm{TV}}$.

Proof. For all $\mu, \nu \in M_{\mathrm{P}}(\mathscr{S})$, one has

$$\left\|\mu - \nu\right\|_{\mathrm{TV}} = \sup_{A\in\mathscr{S}} \left(\mu(A) - \nu(A)\right) = \sup_{A\in\mathscr{S}} \left(\nu(A) - \mu(A)\right).$$

Therefore,

$$\begin{aligned}
\left\|\mu - 1_D \cdot \mu\right\|_{\mathrm{TV}} &= \sup_{A\in\mathscr{S}} \left((1_D \cdot \mu)(A) - \mu(A)\right) \geq (1_D \cdot \mu)(D) - \mu(D) = \mu(\complement D),\\
\left\|\mu - 1_D \cdot \mu\right\|_{\mathrm{TV}} &\leq \sup_{A\in\mathscr{S}} \left(\mu(A\cap D) - (1_D \cdot \mu)(A)\right) + \sup_{A\in\mathscr{S}} \left(\mu(A \cap \complement D)\right) = \mu(\complement D),
\end{aligned}$$

which yields (i). To prove (ii), we observe that

$$\left\|\mu - \nu\right\|_{\mathrm{TV}} = \sup_{0\leq\varphi\leq 1} \left(\mu\varphi - \nu\varphi\right) = \sup_{0\leq\varphi\leq 1} \left(\nu\varphi - \mu\varphi\right).$$

If $\mu(\psi)/\nu(\psi) \geq 1$, then

$$\begin{aligned}
\left\|\psi \cdot \mu - \psi \cdot \nu\right\|_{\mathrm{TV}} &= \frac{\left\|\psi\right\|_\infty}{\mu(\psi)} \sup_{0\leq\varphi\leq 1} \left(\frac{\mu(\psi\varphi)}{\left\|\psi\right\|_\infty} - \frac{\mu(\psi)}{\nu(\psi)} \frac{\nu(\psi\varphi)}{\left\|\psi\right\|_\infty}\right)\\
&\leq \frac{\left\|\psi\right\|_\infty}{\mu(\psi)} \sup_{0\leq\varphi\leq 1} \left(\frac{\mu(\psi\varphi)}{\left\|\psi\right\|_\infty} - \frac{\nu(\psi\varphi)}{\left\|\psi\right\|_\infty}\right) \leq \frac{\left\|\psi\right\|_\infty}{\mu(\psi)} \left\|\mu - \nu\right\|_{\mathrm{TV}}
\end{aligned}$$

If $\mu(\psi)/\nu(\psi) \leq 1$

$$\begin{aligned}
\left\|\psi \cdot \mu - \psi \cdot \nu\right\|_{\mathrm{TV}} &= \frac{\left\|\psi\right\|_\infty}{\mu(\psi)} \sup_{0\leq\varphi\leq 1} \left(\frac{\mu(\psi)}{\nu(\psi)} \frac{\nu(\psi\varphi)}{\left\|\psi\right\|_\infty} - \frac{\mu(\psi\varphi)}{\left\|\psi\right\|_\infty}\right)\\
&\leq \frac{\left\|\psi\right\|_\infty}{\mu(\psi)} \sup_{0\leq\varphi\leq 1} \left(\frac{\nu(\psi\varphi)}{\left\|\psi\right\|_\infty} - \frac{\mu(\psi\varphi)}{\left\|\psi\right\|_\infty}\right) \leq \frac{\left\|\psi\right\|_\infty}{\mu(\psi)} \left\|\mu - \nu\right\|_{\mathrm{TV}}.
\end{aligned}$$

□

2.2.4 Convergence almost surely and in the mean sense

In the filtering context the convergence is considered for random, rather than deterministic, probability measures. The weak convergence and the convergence in total variation defined above can be naturally extended for random measures by saying that (2.7) or (2.9) holds P-almost surely. Another way to take the randomness of the measures into account in the convergence is to consider the convergence of the expected distance between measures, i.e.

$$\lim_{i\to\infty} \mathsf{E}\left[d_{\mathrm{w}}(\mu_i, \mu)\right] = 0 \tag{2.10}$$

$$\lim_{i\to\infty} \mathsf{E}\left[\left\|\mu_i - \mu\right\|_{\mathrm{TV}}\right] = 0. \tag{2.11}$$

Because d_{w} and $\|\cdot\|_{\mathrm{TV}}$ take values in $[0,1]$, it follows by the dominated convergence theorem that if (2.7) or (2.9) holds P-a.s., then (2.10) or (2.11) holds, respectively. On the other hand, it can be shown that for all $q > 0$ and $p > 1$,

$$\lim_{i\to\infty} i^{q+p}\mathsf{E}\left[d_{\mathrm{w}}(\mu_i,\mu)\right] = 0 \implies \lim_{i\to\infty} i^q d_{\mathrm{w}}(\mu_i,\,\mu) \overset{\text{P-a.s.}}{=} 0,$$

and similarly for the total variation distance

$$\lim_{i\to\infty} i^{q+p}\mathsf{E}\left[\left\|\mu_i - \mu\right\|_{\mathrm{TV}}\right] = 0 \implies \lim_{i\to\infty} i^q \left\|\mu_i - \mu\right\|_{\mathrm{TV}} \overset{\text{P-a.s.}}{=} 0.$$

In other words, the almost sure convergence of probability measures implies the convergence in the mean sense and a sufficiently fast convergence in the mean sense implies almost sure convergence.

We also consider a convergence of the form

$$\lim_{i\to\infty} \mathsf{E}\left[\frac{1}{2}\sup_{\|\varphi\|_\infty\le 1} \mathsf{E}\left[\left|\mu_i\varphi - \mu\varphi\right| \,\middle|\, \mathscr{G}\right]\right] = 0, \tag{2.12}$$

where $\mathscr{G} \subset \mathscr{F}$. This form of convergence is found to be stronger than (2.10) by observing that

$$\begin{aligned}\mathsf{E}\left[d_{\mathrm{w}}(\mu_i,\mu)\right] &= \mathsf{E}\left[\mathsf{E}\left[d_{\mathrm{w}}(\mu_i,\mu)\,\middle|\,\mathscr{G}\right]\right] = \mathsf{E}\left[\frac{1}{2}\sum_{j=1}^{\infty}\frac{\mathsf{E}\left[\left|\mu_i\hat{\varphi}_j - \mu\hat{\varphi}_j\right|\,\middle|\,\mathscr{G}\right]}{2^j\left\|\hat{\varphi}_j\right\|_\infty}\right] \\ &\le \mathsf{E}\left[\frac{1}{2}\sup_{\|\varphi\|_\infty\le 1}\mathsf{E}\left[\left|\mu_i\varphi - \mu\varphi\right|\,\middle|\,\mathscr{G}\right]\right].\end{aligned} \tag{2.13}$$

The convenience of (2.12) is that it is independent of the choice of $\hat{\Phi}$ and therefore it can be used for obtaining more practical results, as will be seen later in Section 3.2. Moreover, it is observed that

$$\frac{1}{2}\sup_{\|\varphi\|_\infty\le 1}\mathsf{E}\left[\left|\mu_i\varphi - \mu\varphi\right|\,\middle|\,\mathscr{G}\right] \le \mathsf{E}\left[\left\|\mu_i\varphi - \mu\varphi\right\|_{\mathrm{TV}}\,\middle|\,\mathscr{G}\right],$$

and hence (2.11) is also a stronger form of convergence than (2.12). In conclusion, the strongest form of convergence described above is the almost sure convergence in total variation and therefore it should be of primary interest. However, in certain occasions convergence results can only be obtained in a weaker sense, as will be seen later e.g. in Section 3.1.

2.3 Approximation by truncation

Having established the exact filter recursion and the required concepts of convergence, we are now ready to start constructing filter approximations that play an important role both in the proof of the filter stability and in the proof of the uniform convergence. First we define a method for approximating π which is based on truncating the support of the filtering distribution. At the end of this section, the resulting truncated approximation is motivated by describing the principles of how it can be applied to the filter stability analysis.

2.3.1 Truncated filter

Throughout the remainder of this work it is assumed that $S = \mathbb{R}^{d_s}$ and that for all $x \in \mathbb{R}^{d_s}$, $K_i(x, \cdot)$ has a positive density $k_i(x, \cdot)$ with respect to λ_{d_s} such that for all $i > 0$,

$$\left\|k_i\right\|_\infty \triangleq \sup_{x,y\in\mathbb{R}^{d_s}} k_i(x, y) < \infty.$$

For all $\Delta > 0$ we define a $M_P(\mathscr{B}(\mathbb{R}^{d_s}))$-valued stochastic process $\pi^\Delta = (\pi_i^\Delta)_{i\geq 0}$ by the recursion

$$\pi_i^\Delta \triangleq g_i^\Delta \cdot \pi_{i-1}^\Delta K_i, \tag{2.14}$$

where $\pi_0^\Delta = \pi_0$, $g_i^\Delta \triangleq 1_{C_i(\Delta)} g_i$ and $C_i(\Delta) \in \mathscr{B}(\mathbb{R}^{d_s})$ is compact. Obviously, π and π^Δ are not equal in general and therefore π^Δ is called *the truncated approximation* of π with parameter Δ which is referred to as *the truncation radius*. At this point, the specific construction of the sets $C_i(\Delta)$ is not of interest and therefore the detailed specification of $C_i(\Delta)$ is left for Section 2.4. Also note that if $C_i(\Delta) = \mathbb{R}^{d_s}$ then π^Δ equals the exact filter.

For all $i > 0$, we let $Q_i^\Delta : M_P(\mathscr{B}(\mathbb{R}^{d_s})) \to M_P(\mathscr{B}(\mathbb{R}^{d_s}))$ denote the mapping $\pi_{i-1}^\Delta \mapsto \pi_i^\Delta$. Moreover, for all $i \geq j \geq 0$, we define $Q_{j,i}^\Delta \triangleq Q_i^\Delta \circ \cdots \circ Q_j^\Delta$ and $Q_{j,i} \triangleq Q_i \circ \cdots \circ Q_j$, where Q_0^Δ and Q_0 are assumed to be identity mappings. Also, for all $i < j$, $Q_{j,i}^\Delta$ and $Q_{j,i}$ are defined to be identity mappings. In this case, we further define a mapping $\Pi_{i,j}^\Delta : M_P(\mathscr{B}(\mathbb{R}^{d_s})) \to M_P(\mathscr{B}(\mathbb{R}^{d_s}))$ as

$$\Pi_{i,j}^\Delta(\mu) \triangleq Q_{j+1,i}^\Delta(Q_{1,j}(\mu)). \tag{2.15}$$

Essentially, (2.15) defines for all $j \geq 0$ a probability measure valued stochastic process $(\Pi_{i,j}^\Delta(\mu))_{i\geq 0}$ starting from $\mu \in M_P(\mathscr{B}(\mathbb{R}^{d_s}))$. An intuitive interpretation for these processes is that for the jth process, the j first steps of recursion are exact and the remaining steps are approximate. In particular, we use the shorthand notations $\pi_{i,j}^\Delta \triangleq \Pi_{i,j}^\Delta(\pi_0)$ and $\bar{\pi}_{i,j}^\Delta \triangleq \Pi_{i,j}^\Delta(\bar{\pi}_0)$, where $\bar{\pi}_0 \in M_P(\mathscr{B}(\mathbb{R}^{d_s}))$ is arbitrary. Clearly, $\pi_{i,0}^\Delta = \pi_i^\Delta$ and $\pi_{i,i}^\Delta = \pi_i$ and therefore we define analogously $\bar{\pi}_i \triangleq \bar{\pi}_{i,i}^\Delta$ and $\bar{\pi}_i^\Delta \triangleq \bar{\pi}_{i,0}^\Delta$. An illustration of the notation is given in

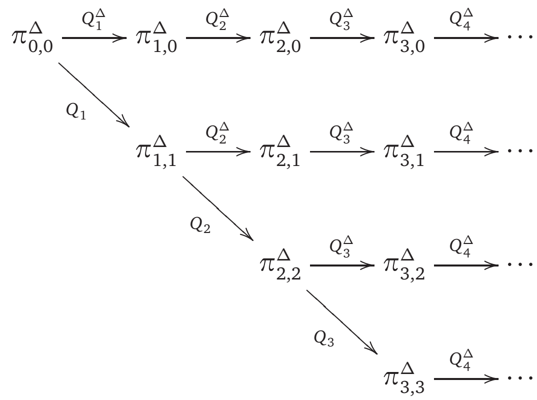

Figure 2.1: The exact filter process and approximations based on truncation.

Figure 2.1. It is important to point out that although $\Pi^\Delta_{i,i}(\pi_0)$, according to Theorem 2.1, is the conditional distribution of the signal X with initial distribution π_0 at time i, $\Pi^\Delta_{i,i}(\mu)$ is not in general the conditional distribution of the signal X with initial distribution μ at time i. This is because the mappings $\Pi^\Delta_{i,j}$ depend on Y through the likelihood functions g_{i,Y_i} and regardless of the choice of μ, Y is always assumed to be determined by the observation noise V and the signal X which is assumed to have the initial distribution π_0. Hereafter, the process $(\Pi^\Delta_{i,i}(\mu))_{i\geq 0}$, regardless of the choice of μ, will be called the exact filter with initial distribution μ. It should not, however, be confused with the filter π by which we mean the process $(\Pi^\Delta_{i,i}(\pi_0))_{i\geq 0}$.

Let us then derive an alternative formulation of the processes $(\Pi^\Delta_{i,j}(\mu))_{i\geq 0}$, $j \geq 0$. For all $i \geq j > 0$ and $\Delta > 0$, we define a mapping $S^\Delta_{j,i} : \mathbb{R}^{d_s} \times \mathscr{B}(\mathbb{R}^{d_s}) \to [0,1]$, such that for all $x \in \mathbb{R}^{d_s}$ and $A \in \mathscr{B}(\mathbb{R}^{d_s})$

$$S^\Delta_{j,i}(x,A) \triangleq \frac{K_j(g^\Delta_j K_{j+1}(g^\Delta_{j+1} \cdots K_i(g^\Delta_i))1_A)(x)}{K_j(g^\Delta_j K_{j+1}(g^\Delta_{j+1} \cdots K_i(g^\Delta_i)))(x)},$$

and a mapping $\psi^\Delta_{j,i} : \mathbb{R}^{d_s} \to \mathbb{R}$ such that for all $x \in \mathbb{R}^{d_s}$

$$\psi^\Delta_{j,i}(x) \triangleq \begin{cases} K_j(g^\Delta_j K_{j+1}(g^\Delta_{j+1} \cdots K_i(g^\Delta_i)))(x) & \text{if } i \geq j \\ 1 & \text{if } i < j. \end{cases}$$

An alternative representation of $(\Pi^\Delta_{i,j}(\mu))_{i\geq 0}$ is then obtained according to the following lemma. This representation was originally proposed in [22] and it also appears in [56].

Lemma 2.4. *For all $i \geq j > 0$ and $\mu \in M_{\mathrm{P}}(\mathscr{B}(\mathbb{R}^{d_s}))$,*

$$Q^{\Delta}_{j,i}(\mu) = (\psi^{\Delta}_{j,i} \cdot \mu) S^{\Delta}_{j,i} S^{\Delta}_{j+1,i} \cdots S^{\Delta}_{i,i}. \tag{2.16}$$

Proof. For all $i \geq j > 0$

$$\begin{aligned} Q^{\Delta}_{j,i}(\mu)(\varphi) &= \frac{\mu(K_j(g^{\Delta}_j \cdots K_i(g^{\Delta}_i \varphi)))}{\mu(K_j(g^{\Delta}_j \cdots K_i(g^{\Delta}_i)))} \\ &= \frac{1}{\mu(\psi^{\Delta}_{j,i})} \mu\left(\frac{\psi^{\Delta}_{j,i}}{\psi^{\Delta}_{j,i}} K_j \left(\frac{g^{\Delta}_j \psi^{\Delta}_{j+1,i}}{\psi^{\Delta}_{j+1,i}} K_{j+1} \left(\cdots \frac{g^{\Delta}_{i-1} \psi^{\Delta}_{i,i}}{\psi^{\Delta}_{i,i}} K_i \left(g^{\Delta}_i \psi^{\Delta}_{i+1,i} \varphi \right) \right) \right) \right). \end{aligned}$$

The substitution of

$$S^{\Delta}_{j,i}(\varphi)(x) = \frac{1}{\psi^{\Delta}_{j,i}(x)} K_j(g^{\Delta}_j \psi^{\Delta}_{j+1,i} \varphi)(x),$$

yields

$$Q^{\Delta}_{j,i}(\mu)(\varphi) = \frac{1}{\mu(\psi^{\Delta}_{j,i})} \mu(\psi^{\Delta}_{j,i} S^{\Delta}_{j,i}(S^{\Delta}_{j+1,i}(\cdots S^{\Delta}_{i,i}(\varphi)))) = \left((\psi^{\Delta}_{j,i} \cdot \mu) S^{\Delta}_{j,i} S^{\Delta}_{j+1,i} \cdots S^{\Delta}_{i,i} \right)(\varphi). \quad \square$$

The representation given by Lemma 2.4 is motivated by the observation that the mapping $\mu \mapsto \mu K$ is a contraction with respect to the total variation distance. The contraction coefficient equals $1 - \alpha_S(K)$, where $\alpha_S(K) \in [0,1]$ is the Dobrushin ergodic coefficient and it is defined as

$$\alpha_S(K) \triangleq 1 - \sup_{\substack{x,y \in S \\ A \in \mathscr{S}}} \left| K(x,A) - K(y,A) \right|.$$

The proof of the contractivity is given in [25, 26]. Consequently, we can bound the error of two differently initialised truncated approximation in the total variation distance, according to the following result. Note that in the following lemma and throughout the remainder of this work, the products over empty sets are considered to be equal to one.

Lemma 2.5. *For all $i, j > 0$ and $\mu, \mu' \in M_{\mathrm{P}}(\mathscr{B}(\mathbb{R}^{d_s}))$*

$$\left\| Q^{\Delta}_{j,i}(\mu) - Q^{\Delta}_{j,i}(\mu') \right\|_{\mathrm{TV}} \leq \prod_{n=j+1}^{i} \left(1 - \alpha_{C_{n-1}(\Delta)}(S^{\Delta}_{n,i}) \right) \left\| \psi^{\Delta}_{j,i} \cdot \mu - \psi^{\Delta}_{j,i} \cdot \mu' \right\|_{\mathrm{TV}}.$$

Proof. If $j > i$, the claim is trivial. For all $i \geq j > 0$, according to Lemma 2.4,

$$\left\| Q^{\Delta}_{j,i}(\mu) - Q^{\Delta}_{j,i}(\mu') \right\|_{\mathrm{TV}} = \left\| (\psi^{\Delta}_{j,i} \cdot \mu) S^{\Delta}_{j,i} \cdots S^{\Delta}_{i,i} - (\psi^{\Delta}_{j,i} \cdot \mu') S^{\Delta}_{j,i} \cdots S^{\Delta}_{i,i} \right\|_{\mathrm{TV}}.$$

For all $j < n \leq i$ the transition probabilities $S^{\Delta}_{n,i}$ are applied to a probability measure on

$C_{n-1}(\Delta)$, and $S^{\Delta}_{j,i}$ is applied to a probability measure on $\mathbb{R}^{d_s}$. Thus,

$$\left\|Q^{\Delta}_{j,i}(\mu) - Q^{\Delta}_{j,i}(\mu')\right\|_{\mathrm{TV}} \leq \left(1 - \alpha_{\mathbb{R}^{d_s}}(S^{\Delta}_{j,i})\right) \prod_{n=j+1}^{i} \left(1 - \alpha_{C_{n-1}(\Delta)}(S^{\Delta}_{n,i})\right) \left\|\psi^{\Delta}_{j,i} \cdot \mu - \psi^{\Delta}_{j,i} \cdot \mu'\right\|_{\mathrm{TV}},$$

from which the claim follows because $\alpha_{\mathbb{R}^{d_s}}(S^{\Delta}_{j,i}) \in [0,1]$. □

For the stability and convergence analysis, it is natural that $S^{\Delta}_{j,i}$ should be as contractive as possible. In order to bound this contractivity, we need to bound the ergodic coefficients $\alpha_{C_{j-1}(\Delta)}(S^{\Delta}_{j,i})$ from below. For this purpose, we define

$$\tilde{\alpha}_i(\Delta) \triangleq \left\|k_i\right\|_{\infty}^{-1} \inf_{\substack{y \in C_i(\Delta) \\ x \in C_{i-1}(\Delta)}} k_i(x,y), \tag{2.17}$$

for which we have the following result from [22].

Lemma 2.6. *For all $i \geq j > 0$,*

$$\alpha_{C_{j-i}(\Delta)}(S^{\Delta}_{j,i}) \geq \tilde{\alpha}_j(\Delta)$$

Proof. For all $i \geq j > 0$, $x \in C_{j-1}(\Delta)$ and $A \in \mathscr{B}(\mathbb{R}^{d_s})$

$$\begin{aligned} S^{\Delta}_{j,i}(x,A) &= \frac{K_j(g^{\Delta}_j \psi^{\Delta}_{j+1,i} 1_A)(x)}{K_j(g^{\Delta}_j \psi^{\Delta}_{j+1,i})(x)} = \frac{\lambda_{d_s}(k_j(x,\cdot) g^{\Delta}_j \psi^{\Delta}_{j+1,i} 1_A)}{\lambda_{d_s}(k_j(x,\cdot) g^{\Delta}_j \psi^{\Delta}_{j+1,i})} \\ &\geq \frac{\inf_{\substack{y \in C_j(\Delta) \\ x \in C_{j-1}(\Delta)}} k_j(x,y)}{\sup_{\substack{y \in C_j(\Delta) \\ x \in C_{j-1}(\Delta)}} k_j(x,y)} \frac{\lambda_{d_s}(g^{\Delta}_j \psi^{\Delta}_{j+1,i} 1_A)}{\lambda_{d_s}(g^{\Delta}_j \psi^{\Delta}_{j+1,i})} \end{aligned}$$

and thus

$$\alpha_{C_{j-1}(\Delta)}(S^{\Delta}_{j,i}) = \inf \sum_{n=1}^{M} \min\left(S^{\Delta}_{j,i}(x,A_n), S^{\Delta}_{j,i}(y,A_n)\right) \geq \tilde{\alpha}_j(\Delta),$$

where the first equality is equivalent to the definition of the Dobrushin ergodic coefficient (see Eq. (1.16) and Eq. (1.5‴) in [25] and Section 3.2 in [26], see also Eq. (6) in [22]). The infimum is taken over all $x, y \in C_{j-1}(\Delta)$ and all M set partitions of $\mathbb{R}^{d_s}$. □

According to the elementary inequality $\prod_i (1 - a_i) \leq \exp\left(-\sum_i a_i\right)$ which holds for all $a_i \in [0,1]$, one has

$$\lim_{i \to \infty} \prod_{n=j+1}^{i} \left(1 - \alpha_{C_{n-1}(\Delta)}(S^{\Delta}_{n,i})\right) \leq \exp\left(-\lim_{i \to \infty} \sum_{n=j+1}^{i} \tilde{\alpha}_n(\Delta)\right).$$

Therefore, if $C_i(\Delta) = S$ for all $i \geq 0$, then $Q_i^\Delta = Q_i$ for all $i \geq 0$ as well and it follows directly from Lemma 2.5 that if $\sum_{i=1}^{\infty} \tilde{\alpha}_i(\Delta) = \infty$, then the filter is stable in the sense that

$$\lim_{i\to\infty} \left\| Q_i \circ \cdots \circ Q_1(\pi_0) - Q_i \circ \cdots \circ Q_1(\bar{\pi}_0) \right\|_{\mathrm{TV}} \overset{\text{P-a.s.}}{=} 0,$$

for all $\bar{\pi}_0 \in M_{\mathrm{P}}(\mathscr{B}(\mathbb{R}^{d_s}))$. This happens if for instance S is compact and k_i is bounded away from zero uniformly with respect to i. A more detailed account on this matter is given in [22, see e.g., Lemma 2.3, Theorem 2.4].

This section is concluded by stating a result which establishes an upper bound for the truncation error, i.e. the distance between the truncated filter π^Δ and the exact filter π. Except for some technical details, the proof can also be found in [56].

Proposition 2.7. *For all $i > 0$,*

$$\left\| \bar{\pi}_i - \bar{\pi}_i^\Delta \right\|_{\mathrm{TV}} \leq \sum_{j=1}^{i} \prod_{n=j+2}^{i} (1 - \tilde{\alpha}_n(\Delta)) \min\left(1, \frac{\left\| \bar{\pi}_{j,j}^\Delta - \bar{\pi}_{j,j-1}^\Delta \right\|_{\mathrm{TV}}}{\tilde{\alpha}_{j+1}(\Delta)}\right). \tag{2.18}$$

Proof. For all $i > 0$,

$$\begin{aligned}
\left\| \bar{\pi}_i - \bar{\pi}_i^\Delta \right\|_{\mathrm{TV}} &\leq \sum_{j=1}^{i} \left\| \bar{\pi}_{i,j}^\Delta - \bar{\pi}_{i,j-1}^\Delta \right\|_{\mathrm{TV}} \\
&\leq \sum_{j=1}^{i} \left\| Q_{j+1,i}^\Delta(\bar{\pi}_{j,j}) - Q_{j+1,i}^\Delta(\bar{\pi}_{j,j-1}) \right\|_{\mathrm{TV}} \\
&\leq \sum_{j=1}^{i} \prod_{n=j+2}^{i} (1 - \tilde{\alpha}_n(\Delta)) \left\| \psi_{j+1,i}^\Delta \cdot \bar{\pi}_{j,j}^\Delta - \psi_{j+1,i}^\Delta \cdot \bar{\pi}_{j,j-1}^\Delta \right\|_{\mathrm{TV}} \\
&\leq \sum_{j=1}^{i} \prod_{n=j+2}^{i} (1 - \tilde{\alpha}_n(\Delta)) \min\left(1, \frac{\left\| \psi_{j+1,i}^\Delta \right\|_\infty \left\| \bar{\pi}_{j,j}^\Delta - \bar{\pi}_{j,j-1}^\Delta \right\|_{\mathrm{TV}}}{\bar{\pi}_{j,j-1}^\Delta(\psi_{j+1,i}^\Delta)}\right),
\end{aligned}$$

where the first inequality is due to the triangle inequality (see Figure 2.1), the second inequality follows from the definition of $\bar{\pi}_{i,j}^\Delta$, the third inequality follows from Lemma 2.5 and the last inequality follows from Lemma 2.3(ii) and the fact that $\|\cdot\|_{\mathrm{TV}}$ is subunitary.

Moreover, it is observed that $\bar{\pi}^{\Delta}_{j,j-1} = 1_{C_j(\Delta)} \cdot \bar{\pi}^{\Delta}_{j,j}$, and hence for all $i \geq j > 0$

$$\begin{aligned}
\frac{\psi^{\Delta}_{j+1,i}(x)}{\bar{\pi}^{\Delta}_{j,j-1}(\psi^{\Delta}_{j+1,i})} &= \frac{K_{j+1}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i})(x)}{\bar{\pi}^{\Delta}_{j,j-1}(1_{C_j(\Delta)}K_{j+1}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i}))} \\
&\leq \frac{\left\|k_{j+1}\right\|_{\infty}}{\inf_{\substack{x\in C_j(\Delta)\\ y\in C_{j+1}(\Delta)}} k_{j+1}(x,y)} \frac{\lambda_{d_s}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i})}{\bar{\pi}^{\Delta}_{j,j-1}(\lambda_{d_s}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i}))} \\
&= \frac{1}{\tilde{\alpha}_{j+1}(\Delta)}.
\end{aligned}$$

□

Note that the bound for the error at time i given by Lemma 2.7 is expressed solely in terms of $\tilde{\alpha}_j(\Delta)$, $1 < j \leq i$, and *the local errors* $\left\|\bar{\pi}^{\Delta}_{j,j} - \bar{\pi}^{\Delta}_{j,j-1}\right\|$, $0 < j \leq i$. Moreover, Proposition 2.7 yields the following less complicated corollary.

Corollary 2.8. *For all $i > 0$,*

$$\left\|\bar{\pi}_i - \bar{\pi}^{\Delta}_i\right\|_{\text{TV}} \leq \sum_{j=1}^{i} \frac{\left\|\bar{\pi}^{\Delta}_{j,j} - \bar{\pi}^{\Delta}_{j,j-1}\right\|_{\text{TV}}}{\tilde{\alpha}_{j+1}(\Delta)}. \tag{2.19}$$

Proof. Follows directly from Proposition 2.7, because $(1 - \tilde{\alpha}_j(\Delta)) \in [0, 1]$. □

2.3.2 Application to the stability analysis

Let us briefly describe the principle of how the truncated filter can be applied to proving the stability of the filter. By the triangle inequality

$$\left\|\pi_i - \bar{\pi}_i\right\|_{\text{TV}} \leq \left\|\pi_i - \pi^{\Delta}_i\right\|_{\text{TV}} + \left\|\pi^{\Delta}_i - \bar{\pi}^{\Delta}_i\right\|_{\text{TV}} + \left\|\bar{\pi}^{\Delta}_i - \bar{\pi}_i\right\|_{\text{TV}}. \tag{2.20}$$

Based on the numerous results on the filter stability in the case of compact state space, it is natural to conjecture that under some assumptions the second term on the right hand side of (2.20) converges to zero as $i \to \infty$. A rigorous proof of this intuitive statement is given later in Proposition 2.14. Moreover, it is intuitive that the remaining two terms in the right hand side of (2.20) do not in general converge to zero as $i \to \infty$. In fact, it can be shown that in some cases these terms converge to one as $i \to \infty$. For details, see Example 3.1 in Section 3.1.

Because the error due to the truncation is expected to decrease as the set $C_i(\Delta)$ is made larger, $C_i(\Delta)$ will be parameterised by Δ such that for all $\Delta < \Delta'$, $C_i(\Delta) \subset C_i(\Delta')$. Also, instead of (2.20) we consider the decomposition

$$\left\|\pi_i - \bar{\pi}_i\right\|_{\mathrm{TV}} \leq \left\|\pi_i - \pi_i^{\Delta_i}\right\|_{\mathrm{TV}} + \left\|\pi_i^{\Delta_i} - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}} + \left\|\bar{\pi}_i^{\Delta_i} - \bar{\pi}_i\right\|_{\mathrm{TV}}, \tag{2.21}$$

where $\Delta_i \leq \Delta_{i+1}$ for all $i > 0$. It can be shown under some assumptions that if $(\Delta_i)_{i>0}$ is a sufficiently fast increasing sequence, then the first and the last terms on the right hand side of (2.21) converge to zero as $i \to \infty$. On the other hand, it may be the case that if $(\Delta_i)_{i>0}$ increases too fast, the convergence of the middle term does not hold. Therefore the proof of the filter stability boils down to proving that there exists a rate for $(\Delta_i)_{i>0}$ such that all terms on the right hand side of (2.21) converge to zero.

In order to avoid confusion, it should be emphasised that by definition

$$\pi_i^{\Delta_i} = Q_i^{\Delta_i} \circ Q_{i-1}^{\Delta_i} \circ \cdots \circ Q_1^{\Delta_i}(\pi_0),$$

which in general is not equal to $Q_i^{\Delta_i} \circ Q_{i-1}^{\Delta_{i-1}} \circ \cdots \circ Q_1^{\Delta_1}(\pi_0)$. In other words, $\pi_i^{\Delta_i}$ is obtained by using the same truncation radius Δ_i in all steps of the recursion up to time i and not by using Δ_j in the jth step of the recursion for all $0 < j \leq i$.

2.4 Filter framework specification

So far, we have considered a rather general filter framework. In the following, the filter framework is further specified by introducing the following assumptions:

(A1) For all $i > 0$, the signal process X satisfies

$$X_i = f_i(X_{i-1}) + W_i,$$

where W_i is an independent random variable with a distribution $\mathsf{P}_{W_i} \in M_{\mathrm{P}}(\mathscr{B}(\mathbb{R}^{d_s}))$ and $f_i : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ is continuous. Moreover, there exist $\alpha > 0$ and $\delta \geq 0$ such that

$$\left\|f_i(x) - f_i(y)\right\| \leq \alpha \left\|x - y\right\| + \delta$$

for all $x, y \in \mathbb{R}^{d_s}$ and $i > 0$.

(A2) There exist $\beta, \beta_0 > 0$, and $\gamma \geq 0$ such that for all $i > 0$, $h_i = \tilde{h}_i + \bar{h}_i$, where $\bar{h}_i : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ is such that $\sup_{x\in\mathbb{R}^{d_s}} \left\|\bar{h}_i(x)\right\| \leq \gamma$ and $\tilde{h}_i : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ is a bijection such that $\tilde{h}_i$ and $\tilde{h}_i^{-1}$ are Lipschitz with coefficients β_0 and β, respectively.

(A3) There exist $m_1, M_1, a_1, A_1, b_1, B_1 > 0$ such that for all $i > 0$

$$m_1 \exp\left(-a_1 \|x\|^{b_1}\right) \leq \rho_{W_i}(x) \leq M_1 \exp\left(-A_1 \|x\|^{B_1}\right),$$

where ρ_{W_i} is the density of P_{W_i} with respect to λ_{d_s}.

(A4) There exist $m_2, M_2, a_2, A_2, b_2, B_2 > 0$ such that for all $i > 0$

$$m_2 \exp\left(-a_2 \|x\|^{b_2}\right) \leq \rho_{V_i}(x) \leq M_2 \exp\left(-A_2 \|x\|^{B_2}\right), \tag{2.22}$$

where ρ_{V_i} is the density of P_{V_i} with respect to λ_{d_m}.

The stochastic process $W = (W_i)_{i>0}$ is called *the signal noise* process and it follows from (A1) and (1.1) that for all $i > 0$ the observation Y_i can be expressed as a function of X_0, V_i and $W_1, W_2, \ldots, W_i$. This implies that Y_i is measurable with respect to the σ–field $\mathscr{F}_i$ which is defined for $i = 0$ as $\mathscr{F}_0 = \sigma(X_0)$ and for all $i > 0$ as

$$\mathscr{F}_i \triangleq \sigma(X_0, V_1, V_2, \ldots, V_i, W_1, W_2, \ldots, W_i).$$

Throughout the remainder of this work it is assumed that (A1), (A2), (A3), and (A4) are satisfied by the filter framework under consideration. Note that because of the assumed bijectivity of $\tilde{h}_i$ in (A2), it is required in practice that $d_s = d_m$.

Let us also further specify the truncated approximation π^Δ by defining the sets $C_i(\Delta)$, $i \geq 0$ for all $\Delta > 0$ as

$$C_i(\Delta) \triangleq \begin{cases} \left\{x \in \mathbb{R}^{d_s} \;\middle|\; \|Y_i - \tilde{h}_i(x)\| \leq \Delta\right\} & \text{if } i > 0 \\ \left\{x \in \mathbb{R}^{d_s} \;\middle|\; \|x\| \leq \beta\Delta + \beta\gamma\right\} & \text{if } i = 0. \end{cases} \tag{2.23}$$

The interpretation of $C_i(\Delta)$ is that it is the preimage of the Y_i centered ball of radius Δ with respect to $\tilde{h}_i$. Throughout the remainder of this work, the truncated filter π^Δ is assumed to employ this particular definition of $C_i(\Delta)$.

It is natural to conjecture that the error of the truncated approximation π^Δ decreases as $\Delta \to \infty$. This will be rigorously proved later in Lemma 2.13 but at this point it suffices to note that for this reason we are interested in π^Δ only for large values of Δ. Therefore it suffices to consider π^Δ only for $\Delta > \Delta_0$, where it is assumed for convenience that $\Delta_0 > \gamma$.

Under the assumptions given above, a tractable lower bound for $\tilde{\alpha}_i(\Delta)$, and thus for $\alpha_{C_{j-1}(\Delta)}(S^{\Delta}_{j,i})$, can be obtained. For this purpose we define $\varepsilon : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ as

$$\varepsilon(x,y) \triangleq \frac{m_1}{M_1} \exp\left(-a_1(L(x)+y)^{b_1}\right), \tag{2.24}$$

where

$$L(x) = (\alpha\beta + \beta)(x+\gamma) + \delta.$$

Also, for all $i > 0$ we define a random variable ξ_i taking values in $\mathbb{R}_+$ as

$$\xi_i \triangleq \begin{cases} \beta \left\|V_i\right\| + \left\|W_i\right\| + \alpha\beta \left\|V_{i-1}\right\| & \text{if } i > 1 \\ \beta \left\|V_i\right\| + \left\|W_i\right\| + \alpha \left\|X_0\right\| & \text{if } i = 1, \end{cases} \tag{2.25}$$

and a shorthand notation

$$\varepsilon_i(\Delta) \triangleq \varepsilon(\Delta, \xi_i).$$

According to these definitions, we have the following result [see also 56, Lemma 3.3].

Lemma 2.9. *For all $i > 0$, and $\Delta > 0$, one has $\tilde{\alpha}_i(\Delta) \geq \varepsilon_i(\Delta)$.*

Proof. According to (2.23) and (A2), for all $i > 0$, $x_i \in C_i(\Delta)$,

$$\left\|x_i - X_i\right\| \leq \beta \left\|\tilde{h}_i(x_i) - Y_i + \bar{h}_i(X_i) + V_i\right\| \leq \beta\Delta + \beta\gamma + \beta \left\|V_i\right\|.$$

If $i = 0$, then for all $x_0 \in C_0(\Delta)$

$$\left\|x_0 - X_0\right\| \leq \left\|x_0\right\| + \left\|X_0\right\| \leq \beta\Delta + \beta\gamma + \left\|X_0\right\|,$$

Therefore, according to (A1) and (2.25) for all $i > 0$,

$$\begin{aligned} \left\|x_i - f_i(x_{i-1})\right\| &\leq \left\|x_i - X_i\right\| + \left\|f_i(X_{i-1}) + W_i - f_i(x_{i-1})\right\| \\ &\leq \left\|x_i - X_i\right\| + \alpha \left\|x_{i-1} - X_{i-1}\right\| + \delta + \left\|W_i\right\| \\ &\leq (\alpha\beta + \beta)(\Delta + \gamma) + \delta + \xi_i. \end{aligned}$$

The claim then follows by substituting the last form into

$$\tilde{\alpha}_i(\Delta) \geq \frac{m_1}{M_1} \inf_{\substack{y \in C_i(\Delta) \\ x \in C_{i-1}(\Delta)}} \exp\left(-a_1 \left\|y - f_i(x)\right\|^{b_1}\right),$$

which holds by (2.17), (A1) and (A3). □

According to Lemma 2.9, a new bound for the error $\left\|\bar{\pi}_i - \bar{\pi}_i^{\Delta}\right\|_{\mathrm{TV}}$ can be obtained by replacing $\tilde{\alpha}_i(\Delta)$ with $\varepsilon_i(\Delta)$ in Lemma 2.7. Moreover, it follows from Lemma 2.9 that the next step in the course of bounding $\tilde{\alpha}_i(\Delta)$ and $\alpha_{C_{j-1}(\Delta)}(S_{j,i}^{\Delta})$ from below is to find a lower bound for $\varepsilon_i(\Delta)$. For this purpose it is sufficient to bound ξ_i from above. According to the following result such an upper bound can be obtained by using the assumptions (A3) and (A4). It should also be pointed out that at this point the analysis most significantly departs from the analysis given in [56].

Proposition 2.10. *For all $\epsilon \in (0, \min(A_1, A_2))$, there are positive random variables $c_1 = c_1(\epsilon)$, $c_2 = c_2(\epsilon)$, and c_3 such that for all $i > 0$, one has $\xi_i \leq \xi_{i,\epsilon}$,* P-a.s. *where*

$$\xi_{i,\epsilon} \triangleq \left(c_1 + (A_1 - \epsilon)^{-1} \ln i\right)^{1/B_1} + (\alpha\beta + \beta)\left(c_2 + (A_2 - \epsilon)^{-1} \ln i\right)^{1/B_2} + c_3.$$

Proof. Let the densities of the random variables $\left\|W_i\right\|$ and $\left\|V_i\right\|$ be denoted by $\rho_{\|W_i\|}$ and $\rho_{\|V_i\|}$, respectively. According to the assumptions (A3) and (A4), for all $x > 0$

$$\rho_{\|W_i\|}(x) \leq M_1 \tilde{S}_{d_s} x^{d_s - 1} \exp\left(-A_1 x^{B_1}\right),$$
$$\rho_{\|V_i\|}(x) \leq M_2 \tilde{S}_{d_m} x^{d_m - 1} \exp\left(-A_2 x^{B_2}\right),$$

where $\tilde{S}_d$ denotes the surface area of a d dimensional unit sphere. Hence, for all $\epsilon' > 0$

$$\sup_{i>0} \mathsf{E}\left[\exp\left((A_1 - \epsilon')\left\|W_i\right\|^{B_1}\right)\right] \leq M_1 \tilde{S}_{d_s} \int x^{d_s - 1} \exp\left((A_1 - \epsilon')x^{B_1} - A_1 x^{B_1}\right) \mathrm{d}x < \infty.$$

Thus for all $q > 1$

$$\mathsf{E}\left[\sum_{i=1}^{\infty} i^{-q} \exp\left((A_1 - \epsilon')\left\|W_i\right\|^{B_1}\right)\right] \leq \sup_{i>0} \mathsf{E}\left[\exp\left((A_1 - \epsilon')\left\|W_i\right\|^{B_1}\right)\right] \sum_{i=1}^{\infty} i^{-q} < \infty,$$

implying that

$$\sum_{i=1}^{\infty} i^{-q} \exp\left((A_1 - \epsilon')\left\|W_i\right\|^{B_1}\right) < \infty, \qquad \text{P-a.s.}$$

Therefore,

$$c \triangleq \sup_{i>0} i^{-q} \exp\left((A_1 - \epsilon')\left\|W_i\right\|^{B_1}\right) < \infty, \qquad \text{P-a.s.}$$

from which we have for all $\epsilon' \in (0, A_1)$

$$\left\|W_i\right\|^{B_1} \leq \frac{\ln c}{A_1 - \epsilon'} + \frac{q \ln i}{A_1 - \epsilon'}.$$

For all $\epsilon \in (0, A_1)$, one can define $q = (A_1 - \epsilon/2)/(A_1 - \epsilon) > 1$, $\epsilon' = \epsilon/2 \in (0, A_1)$ and

$c_1 = \ln c/(A_1 - \epsilon/2)$. In this case,

$$\left\|W_i\right\|^{B_1} \le c_1 + (A_1 - \epsilon)^{-1} \ln i.$$

The above reasoning can also be applied to $\left\|V_i\right\|$ yielding c_2 and finally by setting $c_3 = \alpha \left\|X_0\right\|$ the claim is found to hold for all $i > 0$. □

So far the interest has been in bounding $\tilde{\alpha}_i(\Delta)$ from below. This has been done in order to control the bounds for the error of the truncated approximation given by Proposition 2.7 and Corollary 2.8. In the next proposition, Lemma 2.9 and Proposition 2.10 are applied to Corollary 2.8 in order to further refine the bound for the truncation error.

Proposition 2.11. *If $b_2 = B_2$, then for all $\epsilon \in (0, \min(a_1, a_2))$, there exists a positive random variable $c_4 = c_4(\epsilon)$, such that for all $\Delta > \Delta_0$ and $i > 0$*

$$\left\|\bar{\pi}_i - \bar{\pi}_i^\Delta\right\|_{\mathrm{TV}} \le c_4 \sum_{j=1}^{i} \frac{\exp\left((-A_2 + \epsilon)(\Delta - \gamma)^{B_2} + 2a_1(L(\Delta) + \xi_{j+1,\epsilon})^{b_1}\right)}{1 - \bar{\pi}_{j-1}(\complement C_{j-1}(\Delta))}, \quad \text{P-a.s.} \tag{2.26}$$

Proof. First, we observe that according to Lemma 2.3(i), for $i > 0$ and $t > 1$

$$\begin{aligned}
\left\|\bar{\pi}_{i,i}^\Delta - \bar{\pi}_{i,i-1}^\Delta\right\|_{\mathrm{TV}} &= \bar{\pi}_i(\complement C_i(\Delta)) \\
&= \frac{\int \left[\int_{\complement C_i(\Delta)} g_i(x_i) k_i(x_{i-1}, x_i)\, \mathrm{d}x_i\right] \bar{\pi}_{i-1}(\mathrm{d}x_{i-1})}{\int \left[\int g_i(x_i) k_i(x_{i-1}, x_i)\, \mathrm{d}x_i\right] \bar{\pi}_{i-1}(\mathrm{d}x_{i-1})} \\
&\le \frac{\left\|k_i\right\|_\infty \int \left[\int_{\complement C_i(\Delta)} g_i(x_i)\, \mathrm{d}x_i\right] \bar{\pi}_{i-1}(\mathrm{d}x_{i-1})}{\int_{C_{i-1}(\Delta)} \left[\int_{C_i(\Delta/t)} g_i(x_i) k_i(x_{i-1}, x_i)\, \mathrm{d}x_i\right] \bar{\pi}_{i-1}(\mathrm{d}x_{i-1})}.
\end{aligned} \tag{2.27}$$

Let us first consider the numerator. According to (A2), the transformation $T(z) = \tilde{h}_i^{-1}(Y_i - z)$ is Lipschitz with coefficient β and therefore λ_{d_s}-almost everywhere differentiable according to Rademacher's theorem [see e.g., 29, Theorem 11.1, Corollary 11.9]. Let JT denote the Jacobian of T. Then, by the change of variables with respect to the transformation T, one has for all $\Delta > \Delta_0$,

$$\begin{aligned}
\int_{\complement C_i(\Delta)} g_i(x_i)\mathrm{d}x_i &= \int_{\|z\| > \Delta} \rho_{V_i}(z - \bar{h}_i(T(z)))\, |JT(z)|\, \mathrm{d}z \\
&\le M_2 \|JT\|_\infty \int_{\|z\| \ge \Delta} \exp\left(-A_2 \left\|z - \bar{h}_i(T(z))\right\|^{B_2}\right) \mathrm{d}z \\
&\le M_2 \|JT\|_\infty \int_{\|z\| \ge \Delta} \exp\left(-A_2(\|z\| - \gamma)^{B_2}\right) \mathrm{d}z
\end{aligned}$$

where the first inequality follows from (A4) and the fact that according to (A2) $\|JT\|_\infty < d_s!\beta^{d_s} < \infty$. Then, by switching to polar coordinates, one can check that for all $\epsilon > 0$, there exists $c = c(\epsilon)$ such that for all $\Delta > \Delta_0$

$$\int_{\complement C_i(\Delta)} g_i(x_i)\mathrm{d}x_i \leq c\exp\Big((-A_2+\epsilon)(\Delta-\gamma)^{B_2}\Big). \tag{2.28}$$

Let us then consider the denominator, for which we have

$$\int_{C_{i-1}(\Delta)}\int_{C_i(\Delta/t)} g_i(y)k_i(x,y)\,\mathrm{d}y\,\bar{\pi}_{i-1}(\mathrm{d}x) \geq \lambda_{d_s}(C_i(\Delta/t))\bar{\pi}_{i-1}(C_{i-1}(\Delta)) \inf_{\substack{y\in C_i(\Delta)\\ z\in C_{i-1}(\Delta)}} k_i(z,y) \inf_{x\in C_i(\Delta)} g_i(x).$$

Moreover, according to (A2) and (A4)

$$\begin{aligned}\inf_{x\in C_i(\Delta/t)} g_i(x) &\geq m_2 \inf_{x\in C_i(\Delta/t)} \exp\Big(-a_2\left\|Y_i-\tilde{h}_i(x)-\bar{h}_i(x)\right\|^{b_2}\Big)\\ &\geq m_2 \inf_{x\in C_i(\Delta/t)} \exp\Big(-a_2\left(\left\|Y_i-\tilde{h}_i(x)\right\|+\left\|\bar{h}_i(x)\right\|\right)^{b_2}\Big)\\ &\geq m_2\exp\Big(-a_2(\Delta/t+\gamma)^{b_2}\Big).\end{aligned} \tag{2.29}$$

The substitution of (2.28) and (2.29) into (2.27) yields

$$\left\|\bar{\pi}^{\Delta}_{i,i}-\bar{\pi}^{\Delta}_{i,i-1}\right\|_{\mathrm{TV}} \leq \frac{c\exp\Big((-A_2+\epsilon)(\Delta-\gamma)^{B_2}+a_2(\Delta/t+\gamma)^{b_2}\Big)}{\tilde{\alpha}_i(\Delta)m_2\lambda_{d_s}(C_i(\Delta/t))(1-\bar{\pi}_{i-1}(\complement C_{i-1}(\Delta)))}. \tag{2.30}$$

It should be pointed out that in order to ensure that the numerator converges to zero as $\Delta\to\infty$ it is required that $B_2 \geq b_2$. On the other hand, (A4) implies $B_2 \leq b_2$ and thus we must have $b_2 = B_2$ which was assumed. Set $t = (2a_2/\epsilon)^{1/B_2}$ which clearly is greater than one for all $\epsilon\in(0,a_2)$. It can be shown that there exists $c' = c'(\epsilon)$ such that

$$\exp\Big((-A_2+\epsilon)(\Delta-\gamma)^{B_2}+a_2(\Delta/t+\gamma)^{b_2}\Big) \leq c'\exp\Big((-A_2+2\epsilon)(\Delta-\gamma)^{B_2}\Big). \tag{2.31}$$

For all $x\in B_{d_s}(\tilde{h}_i^{-1}(Y_i),\Delta/\beta_0)$, according to (A2)

$$\Delta \geq \beta_0\left\|\tilde{h}_i^{-1}(Y_i)-x\right\| \geq \left\|Y_i-\tilde{h}_i(x)\right\|.$$

Consequently $B_{d_s}(\tilde{h}_i^{-1}(Y_i),\Delta/\beta_0)\subset C_i(\Delta)$ and therefore

$$\lambda_{d_s}(C_i(\Delta/t)) \geq \lambda_{d_s}(B_{d_s}(\tilde{h}_i^{-1}(Y_i),\Delta/\beta_0 t)) \geq \lambda_{d_s}(B_{d_s}(0,\Delta_0/\beta_0 t)), \tag{2.32}$$

where the last inequality uses the assumption that $\Delta > \Delta_0$ and the translation invariance of the Lebesgue measure. By applying Lemma 2.9 to $\tilde{\alpha}_i(\Delta)$ and by substituting (2.32) and (2.31) into (2.30), it follows that there exists $c'' = c''(\epsilon)$ such that

$$\left\| \bar{\pi}^{\Delta}_{i,i} - \bar{\pi}^{\Delta}_{i,i-1} \right\|_{\mathrm{TV}} \leq c'' \frac{\exp\left((-A_2 + \epsilon)(\Delta - \gamma)^{B_2} + a_1(L(\Delta) + \xi_i)^{b_1}\right)}{(1 - \bar{\pi}_{i-1}(\complement C_{i-1}(\Delta)))}. \tag{2.33}$$

The claim then follows by substituting this inequality into (2.19) of Corollary 2.8 and by replacing ξ_i and ξ_{i+1} with $\xi_{i+1,\epsilon}$ which, according to Proposition 2.10, is their common upper bound. □

Proposition 2.11 gives a bound for the distance between the exact filter and the truncated approximation, but according to the discussion in Section 2.3.2, we are also interested in the distance between two truncated filters with different initial distributions. For this reason, we have the following result which is adapted from the proof of Proposition 3.1 in [56].

Lemma 2.12. *There exists $r \in (0, \infty)$ such that for all $\Delta > 0$, $i > 0$, $n \geq 0$*

$$\mathsf{E}\left[\prod_{j=1}^{i} (1 - \varepsilon_{n+j}(\Delta)) \,\middle|\, \mathscr{F}_n \right] \leq (1 - \tilde{\varepsilon}(\Delta))^{i-1},$$

where $\tilde{\varepsilon}(\Delta) \triangleq \varepsilon(\Delta, r)/2$.

Proof. Because $\varepsilon_i(\Delta) \in [0,1]$, $i > 0$, the claim holds trivially for $i = 1$. For all $i > 1$ we define the shorthand notations $\tau_i(\Delta) = 1 - \varepsilon_i(\Delta)$ and $\tau(x,y) = 1 - \varepsilon(x,y)$. Because ξ_i is $\mathscr{F}_i$ measurable and $\mathscr{F}_i \subset \mathscr{F}_{i+1}$, $i > 0$, it follows that $\tau_j(\Delta)$ and $\varepsilon_j(\Delta)$ are $\mathscr{F}_i$ measurable for all $i \geq j > 0$. Therefore, for all $i > 1$ and $n \geq 0$

$$\mathsf{E}\left[\prod_{j=n+1}^{n+i} \tau_j(\Delta) \,\middle|\, \mathscr{F}_n \right] = \mathsf{E}\left[\mathsf{E}\left[\tau_{n+i}(\Delta)\tau_{n+i-1}(\Delta) \mid \mathscr{F}_{n+i-2}\right] \prod_{j=n+1}^{n+i-2} \tau_j(\Delta) \,\middle|\, \mathscr{F}_n \right]. \tag{2.34}$$

Because $\tau_i(\Delta) \in [0,1]$, we can write for all $\Delta, x > 0$

$$\begin{aligned}
\tau_{n+i-1}(\Delta)\tau_{n+i}(\Delta) &= \tau_{n+i-1}(\Delta)\tau_{n+i}(\Delta)\left(1_{\{\tau(\Delta,x) \geq \tau_{n+i}(\Delta)\}} + 1_{\{\tau(\Delta,x) < \tau_{n+i}(\Delta)\}}\right) \\
&\leq \tau_{n+i-1}(\Delta)\left(\tau(\Delta,x)1_{\{\tau(\Delta,x) \geq \tau_{n+i}(\Delta)\}} + 1_{\{\tau(\Delta,x) < \tau_{n+i}(\Delta)\}}\right) \\
&= \tau_{n+i-1}(\Delta)\left(\tau(\Delta,x) + (1 - \tau(\Delta,x))1_{\{\tau(\Delta,x) < \tau_{n+i}(\Delta)\}}\right) \\
&\leq \tau_{n+i-1}(\Delta)\tau(\Delta,x) + (1 - \tau(\Delta,x))1_{\{\tau(\Delta,x) < \tau_{n+i}(\Delta)\}}.
\end{aligned}$$

Therefore,

$$\mathsf{E}\left[\tau_{n+i-1}(\Delta)\tau_{n+i}(\Delta) \mid \mathscr{F}_{n+i-2}\right] \leq \tau(\Delta,x)\mathsf{E}\left[\tau_{n+i-1}(\Delta) \mid \mathscr{F}_{n+i-2}\right] + (1-\tau(\Delta,x))p_{n+i}(x), \quad (2.35)$$

where, by the definition of τ

$$p_{n+i}(x) \triangleq \mathsf{P}(\tau_{n+i}(\Delta) > \tau(\Delta,x) \mid \mathscr{F}_{n+i-2}) = \mathsf{P}(\xi_{n+i} > x \mid \mathscr{F}_{n+i-2}) = \mathsf{P}(\xi_{n+i} > x).$$

It follows from (A3) and (A4) that there exists $r \in (0,\infty)$ such that $p_i(r) < 1/4$ for all $i > 0$ and we can define $\tilde{\varepsilon}(\Delta) = \varepsilon(\Delta,r)/2$. In this case, it follows from (2.35) that for all $\Delta > 0$

$$\mathsf{E}\left[\tau_{n+1}(\Delta)\tau_{n+2}(\Delta) \mid \mathscr{F}_n\right] \leq \tau(\Delta,r) + \frac{1}{4}(1-\tau(\Delta,r)) < 1-\tilde{\varepsilon}(\Delta)$$

where the first inequality follows from (2.35) and the second inequality is easily checked to hold according to the definition of $\tilde{\varepsilon}(\Delta)$. Thus the claim holds for $i = 1,2$. To complete the proof, it is assumed that the claim holds for $0 < i < m$. Then, by setting $i = m+1$ the substitution of (2.35) into (2.34) yields

$$\begin{aligned}\mathsf{E}\left[\prod_{j=n+1}^{n+m+1}\tau_j(\Delta)\,\middle|\,\mathscr{F}_n\right] &\leq \tau(\Delta,r)\mathsf{E}\left[\prod_{j=n+1}^{n+m}\tau_j(\Delta)\,\middle|\,\mathscr{F}_n\right] \\ &\quad + \frac{1}{4}(1-\tau(\Delta,r))\mathsf{E}\left[\prod_{j=n+1}^{n+m-1}\tau_j(\Delta)\,\middle|\,\mathscr{F}_n\right] \\ &\leq \tau(\Delta,r)(1-\tilde{\varepsilon}(\Delta))^{m-1} + \frac{1}{4}(1-\tau(\Delta,r))(1-\tilde{\varepsilon}(\Delta))^{m-2} \\ &\leq (1-\tilde{\varepsilon}(\Delta))^m. \qquad \square\end{aligned}$$

In order to see how this result is related to the distance between two truncated filters, it is observed that according to Lemma 2.12

$$\mathsf{E}\left[\prod_{j=1}^{i}(1-\varepsilon_{n+j}(\Delta))\right] \leq (1-\tilde{\varepsilon}(\Delta))^{i-1}, \quad (2.36)$$

holds for all $\Delta > 0$, $n \geq 0$ and $i > 0$. By applying (2.36) and Lemma 2.9 to Lemma 2.5, one has for all $i > 0$

$$\mathsf{E}\left[\left\|\pi_i^{\Delta} - \bar{\pi}_i^{\Delta}\right\|_{\mathrm{TV}}\right] \leq \mathsf{E}\left[\prod_{j=2}^{i}(1-\varepsilon_j(\Delta))\right] \leq (1-\tilde{\varepsilon}(\Delta))^{i-2}. \quad (2.37)$$

In other words, under the assumptions (A1), (A2), (A3), and (A4) the truncated filter is exponentially stable in the mean sense.

2.5 Stability theorem

Because $(\xi_{i,\epsilon})_{i\geq 0}$ is an increasing sequence in $\mathbb{R}$, it follows that the bound for $\left\|\bar{\pi}_i - \bar{\pi}_i^{\Delta}\right\|_{\mathrm{TV}}$ given by Proposition 2.11 is also increasing for a fixed value of Δ. Therefore, following the discussion in Section 2.3.2, we consider a sequence $(\bar{\pi}^{\Delta_j})_{j\geq 0}$ of truncated filters where $\bar{\pi}^{\Delta_j} = (\bar{\pi}_i^{\Delta_j})_{i\geq 0}$ and $(\Delta_j)_{j\geq 0}$ is an increasing sequence in $[\Delta_0, \infty)$. Because $(\xi_{i,\epsilon})_{i\geq 0}$ is nondecreasing, it follows from Proposition 2.11 that a sufficient condition for

$$\lim_{i\to\infty}\left\|\bar{\pi}_i - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}} = 0,$$

is to have

$$\lim_{i\to\infty}\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2}+2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big)\sum_{j=1}^{i}(1-\bar{\pi}_{j-1}(\complement C_{j-1}(\Delta_i)))^{-1} = 0, \tag{2.38}$$

which implies

$$\lim_{i\to\infty}\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2}+2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big) = 0. \tag{2.39}$$

We immediately observe that if $b_1 > B_2$, (2.39) does not hold for any increasing sequence $(\Delta_i)_{i\geq 0}$. This is an important observation because it implies that our approach for proving the filter stability is fruitless for filter frameworks where $b_1 > B_2$. In the case $b_1 \leq B_2$, (2.39) can be shown to hold under some additional conditions.

The following result gives a bound for the distance between the exact filter and the truncated approximations with time dependent truncation radius. Moreover, it follows from the proof that a sufficiently fast convergence in (2.39) implies (2.38).

Lemma 2.13. *If $b_2 = B_2$ and $(\Delta_i)_{i\geq 0}$ is a sequence in $\mathbb{R}_+$ such that*

$$\lim_{i\to\infty}\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2}+2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big) = 0, \quad \text{P-a.s.} \tag{2.40}$$

where $\epsilon \in (0, \min(a_1, a_2, A_1, A_2))$, then there exists a positive random variable $c_5 = c_5(\epsilon, \bar{\pi}_0)$ such that for all $i > 0$

$$\left\|\bar{\pi}_i - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}} \leq c_5 i \exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2}+2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big), \qquad \text{P-a.s.}$$

Proof. If (2.40) holds, then $\lim_{i\to\infty} C_0(\Delta_i) = \mathbb{R}^{d_s}$ and by the continuity of measures

$$\lim_{i\to\infty} \bar{\pi}_0(\complement C_0(\Delta_i)) = 1 - \bar{\pi}_0(\mathbb{R}^{d_s}) = 0.$$

Thus there clearly exists $N_{\epsilon,\bar{\pi}_0} \in \mathbb{N}$, such that for all $i > N_{\epsilon,\bar{\pi}_0}$

$$\bar{\pi}_0(\complement C_0(\Delta_i)) < \frac{\epsilon}{1+\epsilon} \tag{2.41}$$

and

$$c'' \exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2} + 2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big) \le \frac{\epsilon}{(1+\epsilon)^2}, \tag{2.42}$$

where c'' is the constant appearing in (2.33). Then, according to (2.33), for all $0 < j \le i$, where $i > N_{\epsilon,\bar{\pi}_0}$,

$$\begin{aligned}
\bar{\pi}_j(\complement C_j(\Delta_i)) &\le \frac{c''\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2} + a_1(L(\Delta_i)+\xi_j)^{b_1}\Big)}{1-\bar{\pi}_{j-1}(\complement C_{j-1}(\Delta_i))} \\
&\le \frac{c''\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2} + 2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big)}{1-\bar{\pi}_{j-1}(\complement C_{j-1}(\Delta_i))} \\
&\le \frac{\epsilon}{(1+\epsilon)^2(1-\bar{\pi}_{j-1}(\complement C_{j-1}(\Delta_i)))}.
\end{aligned} \tag{2.43}$$

Because (2.41) is equivalent to $1/(1-\bar{\pi}_0(\complement C_0(\Delta_i))) < 1+\epsilon$, it follows that

$$\bar{\pi}_1(\complement C_1(\Delta_i)) \le \frac{\epsilon}{1+\epsilon}.$$

By induction, it then follows that for all $0 < j \le i$, one has $\bar{\pi}_j(\complement C_j(\Delta_i)) \le \epsilon/(1+\epsilon)$, i.e.

$$\frac{1}{1-\bar{\pi}_j(\complement C_j(\Delta_i))} < 1+\epsilon.$$

Therefore, according to Proposition 2.11, for all $i > N_{\epsilon,\bar{\pi}_0}$,

$$\begin{aligned}
\left\|\bar{\pi}_i - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}} &\le c_4 \sum_{j=1}^{i} (1+\epsilon)\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2} + 2a_1(L(\Delta_i)+\xi_{j+1,\epsilon})^{b_1}\Big) \\
&\le c_4(1+\epsilon)i\exp\Big((-A_2+\epsilon)(\Delta_i-\gamma)^{B_2} + 2a_1(L(\Delta_i)+\xi_{i+1,\epsilon})^{b_1}\Big)
\end{aligned} \tag{2.44}$$

which yields the claim. □

Let us consider the case $B_2 = b_1$. In this case, a necessary condition for (2.40) is that

$$\sup_{i\geq 0} \frac{\xi_{i+1,\epsilon}}{\Delta_i} < \infty.$$

According to Proposition 2.10, this implies the existence of $c > 0$ such that $\Delta_i^{B_1} > c\ln(i+1)$, for all $i \geq 0$. Such a constant exists if Δ_i, $i > 0$ is defined to be of the form

$$\Delta_i = (s\ln i + \Delta_0^{B_1})^{1/B_1}, \tag{2.45}$$

where $s > 0$. Thus in order to prove the stability according to the discussion in Section 2.3.2, it is sufficient to show that s can be chosen in such a manner that the convergence in (2.40) is sufficiently fast and that

$$\lim_{i\to\infty} \left\| \pi_i^{\Delta_i} - \bar{\pi}_i^{\Delta_i} \right\|_{\mathrm{TV}} = 0. \tag{2.46}$$

According to the following result, (2.46) holds if s is sufficiently small.

Proposition 2.14. *Suppose that $\Delta_i^{B_1} = s\ln i + \Delta_0^{B_1}$, where $s^{-1} > a_1(\alpha\beta+\beta)^{B_1}$ and $b_1 = B_1$. Then for all $p \in (0, 1 - sa_1(\alpha\beta+\beta)^{b_1})$ and $\epsilon \in (0, m_1/2M_1)$, there exists a positive random variable $c_6 = c_6(\epsilon, p)$ such that*

$$\left\| \pi_i^{\Delta_i} - \bar{\pi}_i^{\Delta_i} \right\|_{\mathrm{TV}} \leq c_6 \exp\left((-m_1/2M_1 + \epsilon) i^p\right), \qquad \text{P-a.s.} \tag{2.47}$$

Proof. By using the inequality $(1-a) \leq \exp(-a)$, it can be shown that for all $i > 1$

$$(1 - \tilde{\varepsilon}(\Delta_i))^{i-2} \leq \exp\left(-\frac{m_1}{2M_1} i^{1-q_i}\right) \exp\left(\frac{m_1}{M_1} i^{-q_i}\right),$$

where

$$q_i = sa_1(\alpha\beta+\beta)^{b_1} \left(1 + \frac{\Delta_0^{B_1}}{s\ln i}\right) \left(1 + \frac{(\alpha\beta+\beta)\gamma + \delta + r}{(\alpha\beta+\beta)\Delta_i}\right)^{b_1}.$$

Because, clearly $q_i > sa_1(\alpha\beta+\beta)^{b_1}$, it follows that

$$\sup_{i>1} \exp\left(\frac{m_1}{M_1} i^{-q_i}\right) < \infty.$$

Also, because $\lim_{i\to\infty} 1 - q_i = 1 - sa_1(\alpha\beta+\beta)^{b_1}$, it follows that there exists $c = c(p) > 0$ such that for all $i > 0$

$$(1 - \tilde{\varepsilon}(\Delta_i))^{i-2} \leq c\exp\left(-\frac{m_1}{2M_1} i^p\right), \tag{2.48}$$

where $0 < p < 1 - sa_1(\alpha\beta+\beta)^{b_1}$. Thus, according to (2.37) and (2.48)

$$\mathsf{E}\left[\left\|\pi_i^{\Delta_i} - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}}\right] \leq (1-\tilde{\varepsilon}(\Delta_i))^{i-2} \leq c\exp\left(-\frac{m_1}{2M_1} i^p\right).$$

Therefore, for all $\epsilon \in (0, m_1/2M_1)$,

$$\mathsf{E}\left[\sum_{i=1}^{\infty} \exp\left(\left(\frac{m_1}{2M_1} - \epsilon\right) i^p\right) \left\|\pi_i^{\Delta_i} - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}}\right] \leq \sum_{i=1}^{\infty} c\exp(-\epsilon i^p) < \infty,$$

and thus there exists a positive random variable c_6 such that

$$c_6 \triangleq \sup_{i>0} \exp\left(\left(\frac{m_1}{2M_1} - \epsilon\right) i^p\right) \left\|\pi_i^{\Delta_i} - \bar{\pi}_i^{\Delta_i}\right\|_{\mathrm{TV}} < \infty \qquad \mathsf{P}\text{-a.s.}$$

□

Before stating the main stability result, we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as

$$\theta(x,y) \triangleq 2\left(y^{1/b_1} + \left(\frac{a_1}{A_1}\right)^{1/b_1} + \frac{a_1^{1/b_1}(\alpha\beta+\beta)}{x^{1/b_1}}\right)^{b_1} - \frac{xy}{a_1(\alpha\beta+\beta)^{b_1}},$$

and $\theta_1 : \mathbb{R}_+ \to \mathbb{R}$ as $\theta_1(x) \triangleq \theta(x,1)$. Clearly θ_1 is continuous, strictly decreasing, and

$$\lim_{x\to 0} \theta_1(x) = -\lim_{x\to\infty} \theta_1(x) = \infty,$$

and therefore we can also define $\kappa_p \triangleq \theta_1^{-1}(-p)$. Moreover, we write

$$\Delta_i(s) \triangleq (s \ln i + \Delta_0^{B_1})^{1/B_1},$$

in order to explicitly illustrate the fact that the truncation radii depend on s.

Theorem 2.15. *If one of the conditions*

(i) $b_1 = B_1 < b_2 = B_2$*;*

(ii) $b_1 = B_1 = b_2 = B_2$*, and* $A_2 > \kappa_1$*,*

holds, then

$$\lim_{i\to\infty} \left\|\pi_i - \bar{\pi}_i\right\|_{\mathrm{TV}} = 0, \qquad \mathsf{P}\text{-a.s.} \tag{2.49}$$

Proof. Let us write,

$$\kappa(\epsilon,i,s) \triangleq (-A_2+\epsilon)\left(1 - \frac{\gamma}{\Delta_i(s)}\right)^{B_2} + 2a_1(\alpha\beta+\beta)^{b_1}\left(1+\frac{\gamma}{\Delta_i(s)}\right)^{b_1}\left(1+\frac{\delta+\xi_{i+1,\epsilon}}{(\alpha\beta+\beta)(\Delta_i(s)+\gamma)}\right)^{b_1}\Delta_i(s)^{b_1-B_2}. \tag{2.50}$$

In this case, according to Proposition 2.14 and Lemma 2.13, it suffices to show that there exists $0 < s < a_1^{-1}(\alpha\beta+\beta)^{-b_1}$ and a sufficiently small $\epsilon > 0$ such that

$$\lim_{i\to\infty} i\exp\Big(\kappa(\epsilon,i,s)\Delta_i(s)^{B_2}\Big) = \lim_{i\to\infty} \exp\left(\left(\frac{\ln i}{\Delta_i(s)^{B_2}} + \kappa(\epsilon,i,s)\right)\Delta_i(s)^{B_2}\right) = 0. \tag{2.51}$$

Because $\sup_{i>0}\Delta_i(s) = \infty$, it suffices to show that

$$\lim_{\epsilon\to 0}\lim_{i\to\infty} \frac{s\ln i}{\Delta_i(s)^{B_2}} + s\kappa(\epsilon,i,s) = \lim_{\epsilon\to 0}\lim_{i\to\infty} \frac{s\ln i}{\left(s\ln i + \Delta_0^{B_1}\right)^{B_2/B_1}} + s\kappa(\epsilon,i,s) < 0. \tag{2.52}$$

From Proposition 2.10 we have

$$\lim_{i\to\infty} \frac{\xi_{i+1,\epsilon}}{(\alpha\beta+\beta)(\Delta_i(s)+\gamma)} = \begin{cases} \dfrac{1}{s^{1/B_1}(A_1-\epsilon)^{1/B_1}(\alpha\beta+\beta)} + \dfrac{1}{s^{1/B_1}(A_2-\epsilon)^{1/B_1}} & \text{if } B_1 = B_2 \\ \dfrac{1}{s^{1/B_1}(A_1-\epsilon)^{1/B_1}(\alpha\beta+\beta)} & \text{if } B_2 > B_1. \end{cases} \tag{2.53}$$

It then follows from (2.50), (2.52), and (2.53), that

$$\lim_{\epsilon\to 0}\lim_{i\to\infty} \frac{s\ln i}{\Delta_i(s)^{B_2}} + s\kappa(\epsilon,i,s) = \begin{cases} 1+\theta(A_2, sa_1(\alpha\beta+\beta)^{b_1}) & \text{if } b_1 = B_2 \\ -sA_2 & \text{if } b_1 < B_2. \end{cases} \tag{2.54}$$

Therefore, (2.52) holds for all $0 < s < a_1^{-1}(\alpha\beta+\beta)^{-b_1}$, if $b_1 < B_2$, which completes the proof for the case (i). Let us then consider the case (ii). Because θ_1 is decreasing and $A_2 > \kappa_1$, one has $\theta_1(A_2) < \theta_1(\kappa_1) = -1$ implying that $\theta(A_2,1)+1 < 0$. Moreover, because $\theta(A_2,\cdot)$ is continuous, there exists $x^* < 1$ such that $\theta(A_2,x^*)+1 < 0$. The inequality (2.52) then holds for $s = x^*/a_1(\alpha\beta+\beta)^{b_1}$. □

It is also possible to establish rates for the convergence in (2.49) according to the following corollary.

Corollary 2.16. *If one of the conditions*

(i) $b_1 = B_1 < b_2 = B_2$ *and* $A_2 > p(a_1^{1/B_1}(\alpha\beta+\beta))^{B_2}$*;*

(ii) $b_1 = B_1 = b_2 = B_2$ *and* $A_2 > \kappa_p$

holds, then there exists a positive random variable c_7 *such that for all* $i > 0$

$$\left\|\pi_i - \bar{\pi}_i\right\|_{\mathrm{TV}} \le c_7 i\exp\left(-p(\ln i)^{B_2/B_1}\right), \qquad \text{P-a.s.} \tag{2.55}$$

Proof. Similarly as in the proof of Theorem 2.15, for all $p > 0$

$$\lim_{\epsilon \to 0} \lim_{i \to \infty} \frac{sp(\ln i)^{B_2/B_1}}{\Delta_i(s)^{B_2}} + s\kappa(\epsilon, i, s) = \begin{cases} p + \theta(A_2, sa_1(\alpha\beta + \beta)^{B_1}) & \text{if } b_1 = B_2 \\ ps^{1-B_2/B_1} - sA_2 & \text{if } b_1 < B_2. \end{cases} \tag{2.56}$$

If $A_2 > \kappa_p$, then $\theta_1(A_2) < \theta_1(\kappa_p) = -p$, implying $\theta(A_2, 1) + p < 0$. By the continuity of $\theta(A_2, \cdot)$, there exists $s < a_1^{-1}(\alpha\beta + \beta)^{-B_1}$ such that $\theta(A_2, sa_1(\alpha\beta + \beta)^{B_1}) + p < 0$. Thus, in the case (i), s can be chosen such that the limit in (2.56) is negative. In the case (ii), one can choose $s \in ((A_2/p)^{-B_1/B_2}, a_1^{-1}(\alpha\beta + \beta)^{-B_1})$ which is nonempty. Also in this case, one can check that the limit in (2.56) is negative. The negativity of this limit implies the existence of $\epsilon > 0$ and $c > 0$ such that for all $i > 0$

$$\exp\Big(\kappa(\epsilon, i, s)\Delta_i(s)^{B_2}\Big) \leq c \exp\Big(-p(\ln i)^{B_2/B_1}\Big).$$

Then, according to Proposition 2.14 and Lemma 2.13, there exist $c_5, \bar{c}_5, c_6 > 0$ such that

$$\begin{aligned} \left\|\pi_i - \bar{\pi}_i\right\|_{\mathrm{TV}} &\leq \left\|\pi_i - \pi_i^{\Delta_i(s)}\right\|_{\mathrm{TV}} + \left\|\pi_i^{\Delta_i(s)} - \bar{\pi}_i^{\Delta_i(s)}\right\|_{\mathrm{TV}} + \left\|\bar{\pi}_i - \bar{\pi}_i^{\Delta_i(s)}\right\|_{\mathrm{TV}} \\ &\leq c_5 i \exp\Big(-p(\ln i)^{B_2/B_1}\Big) + c_6 \exp\big((-m_1/2M_1 + \epsilon)i^q\big) + \bar{c}_5 i \exp\Big(-p(\ln i)^{B_2/B_1}\Big), \end{aligned}$$

where $q \in (0, 1 - sa_1(\alpha\beta + \beta)^{B_1})$, from which the claim follows. □

Note that in the case (ii), the rate of convergence can be written as

$$\left\|\pi_i - \bar{\pi}_i\right\|_{\mathrm{TV}} \leq c_7 i^{-p+1}, \qquad \text{P-a.s.}$$

It should also be noted that regarding the stability, (i) in Corollary 2.16 does not impose any restrictions on A_2 in addition to the positivity. This is because for all $A_2 > 0$, there exists $p > 0$ such that (i) holds, and the right hand side of (2.55) is convergent for all $p > 0$ if $B_2 > B_1$. On the other hand, in the case (ii) the convergence holds only if $p > 1$. Because κ_p as a function of p is continuous and increasing, there exists $p > 1$ such that $A_2 > \kappa_p$ if and only if $A_2 > \kappa_1$, which is consistent with Theorem 2.15.

Chapter 3

Uniform convergence

This chapter establishes sufficient conditions for the uniform convergence of two classes of filter approximations. The convergence is considered in the sense of (2.11) and (2.12). The first class of filter approximations consists only of the truncated filter described in Chapter 2 and the motivation for proving the uniform convergence in this case is purely theoretical. This is because in general, the truncated filter is intractable and therefore it is not of any practical interest. The theoretical significance is due to the fact that if the truncated filter is uniformly convergent, then any uniform approximation of the truncated filter is a uniform approximation of the exact filter as well. Therefore the second class of filter approximations for which the uniform convergence is proved consists of uniformly convergent approximations of the truncated filter that can be parameterised such that the approximation error converges to zero as $\Delta \to \infty$. This class of filter approximations is characterised by a set of properties that the approximating algorithm is assumed to have. These properties are general and they do not imply any specific filter approximation algorithm, but it is shown that for example a certain feasible SIR filter type algorithm has these properties and therefore it is uniformly convergent.

This chapter is organised as follows. In Section 3.1, some preliminary results are given accompanied by a theorem which establishes easily verifiable sufficient conditions for the uniform convergence of the truncated filter with respect to the truncation radius. In Section 3.2, the set of approximating algorithms is specified by introducing some properties that the approximating algorithm is assumed to have. Moreover, the main result on the uniform convergence is stated as well as a practical corollary which establishes the uniform convergence of certain point estimates. In Section 3.3, the set of uniformly convergent filter approximations is exemplified by introducing a feasible SIR filter type algorithm which is shown to satisfy the conditions for uniform convergence. The chapter is concluded in Section 3.4 where the uniform convergence results are illustrated by some computer simulations.

3.1 Uniformly convergent approximation by truncation

The upper bound of the truncation error provided by Proposition 2.11 increases without bound as $i \to \infty$ for a fixed $\Delta > 0$. Therefore Proposition 2.11 cannot be used for proving the uniform convergence of π^Δ. Because of this observation it is natural to ask whether the bound of Proposition 2.11 is unnecessarily loose or is it in fact the case that the approximation error of π^Δ does not converge uniformly to zero as $\Delta \to \infty$. Unfortunately the second alternative appears to be the case in general as illustrated by the following example.

Example 3.1. *Let* $\mathrm{N}\left(x, y\right)$ *denote the normal distribution with mean* x *and covariance* y *and let* σ_∞^2 *denote the posterior variance of a time invariant Kalman filter [see e.g., 2] for the model*

$$X_i = aX_{i-1} + W_i$$
$$Y_i = X_i + V_i,$$

where $W_i \sim \mathrm{N}\left(0, \sigma_\mathrm{S}^2\right)$, $V_i \sim \mathrm{N}\left(0, \sigma_\mathrm{M}^2\right)$ *independently and* $0 < a < 1$. *If* $X_0 \sim \mathrm{N}\left(0, \sigma_\infty^2\right)$, *then for all* $i > 0$

$$\pi_i = \mathrm{N}\left(a\hat{X}_{i-1} + c_\mathrm{K}(Y_i - a\hat{X}_{i-1}), \sigma_\infty^2\right),$$

where $c_\mathrm{K} = (a^2\sigma_\infty^2 + \sigma_\mathrm{S}^2)/(a^2\sigma_\infty^2 + \sigma_\mathrm{S}^2 + \sigma_\mathrm{M}^2)$ *is the time invariant Kalman gain and* $\hat{X}_i$ *is the mean of* π_i. *By defining* $Z_i = Y_i - a\hat{X}_{i-1}$, *it follows that*

$$\left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV} \geq \pi_i(\complement C_i(\Delta)) = \int_{-\infty}^{Y_i-\Delta} \mathrm{d}\pi_i + \int_{Y_i+\Delta}^{\infty} \mathrm{d}\pi_i$$
$$= \Phi(Z_i(1-c_\mathrm{K}) - \Delta; 0, \sigma_\infty^2) - \Phi(Z_i(1-c_\mathrm{K}) + \Delta; 0, \sigma_\infty^2) + 1,$$

where $\Phi(\cdot\,; x, y)$ *denotes the distribution function of* $\mathrm{N}\left(x, y\right)$. *Clearly, for all* $\Delta > 0$

$$\lim_{|z|\to\infty} \Phi(z(1-c_\mathrm{K}) - \Delta; 0, \sigma_\infty^2) - \Phi(z(1-c_\mathrm{K}) + \Delta; 0, \sigma_\infty^2) = 0$$

and therefore, for all $0 < \epsilon < 1$, *there exists* $z^*(\epsilon) > 0$, *such that* $\left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV} > 1 - \epsilon$, *whenever* $\left|Z_i\right| > z^*(\epsilon)$. *In the literature, the random variable* Z_i *is commonly referred to as* the innovation *and it can be shown that* Z_i *are mutually independent random variables with* *common distribution* $\mathrm{N}\left(0, a^2\sigma_\infty^2 + \sigma_\mathrm{S}^2 + \sigma_\mathrm{M}^2\right)$. *Therefore,*

$$\mathsf{P}\left(\left|Z_i\right| > z^*(\epsilon)\right) = 2\Phi(-z^*(\epsilon); 0, a^2\sigma_\infty^2 + \sigma_\mathrm{S}^2 + \sigma_\mathrm{M}^2) > 0,$$

and it follows by the Borel-Cantelli lemma that almost surely $\left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV} > 1-\epsilon$ *for infinitely many* $i > 0$*. This implies that*

$$\sup_{i>0} \left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV} > 1 - \epsilon, \qquad \mathsf{P}\text{-a.s.}$$

Because this holds for all $0 < \epsilon < 1$*, it follows that*

$$\sup_{i>0} \left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV} = 1, \qquad \mathsf{P}\text{-a.s.}$$

From this we conclude that π^Δ *cannot converge almost surely to* π *in a uniform manner.*

The preceding example shows that in general π^Δ does not satisfy

$$\lim_{\Delta\to\infty} \sup_{i>0} \left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV} = 0, \qquad \mathsf{P}\text{-a.s.}, \tag{3.1}$$

but it will be shown later by Proposition 3.4 that under some assumptions

$$\lim_{\Delta\to\infty} \sup_{i>0} \mathsf{E}\left[\left\|\pi_i - \pi_i^\Delta\right\|_\mathrm{TV}\right] = 0 \tag{3.2}$$

holds. Therefore the significance of Example 3.1 is twofold. Firstly, it shows that the almost sure uniform convergence in (3.1) is genuinely stronger than the uniform convergence in (3.2) in the sense that by the dominated convergence theorem (3.1) implies (3.2) but according to Example 3.1 the converse implication does not hold. Secondly, it shows the futility of trying to find room for improvements in the proof of (3.2) in order to establish (3.1).

Let us then turn to the proof of (3.2). Similarly as the proof of the filter stability, the proof of (3.2) is based on Proposition 2.7 and bounding $\tilde{\alpha}_i(\Delta)$ from below by $\varepsilon(\Delta, \xi_i)$ but otherwise the approach is somewhat different. In the previous chapter ξ_i was bounded from above according to Proposition 2.10 which gave an upper bound for $\epsilon(\Delta, \xi_i)$ in the almost sure sense. Here, instead of bounding ξ_i we derive upper bounds for the tails of the distribution of ξ_i. This enables us to bound the expectations of nonnegative functions of ξ_i from above. It should be pointed out that the analysis in [56] is similarly based on bounding the distribution of ξ_i instead of using the almost sure bounds for ξ_i. Therefore many similarities between the following analysis and [56] can be found. Let us start with the following result.

Lemma 3.2. *Suppose that a_{A}, a_{B}, b_{A}, $b_{\mathrm{B}} > 0$. If $f_{\mathrm{A}}, f_{\mathrm{B}} : \mathbb{R} \to \mathbb{R}_+$ satisfy $f_{\mathrm{A}}(x) = f_{\mathrm{B}}(x) = 0$ for all $x < 0$ and*

$$\begin{aligned} &\sup_{x\geq 0} f_{\mathrm{A}}(x)\exp\left(ax^{b_{\mathrm{A}}}\right) < \infty \\ &\sup_{x\geq 0} f_{\mathrm{B}}(x)\exp\left(a'x^{b_{\mathrm{B}}}\right) < \infty \end{aligned} \tag{3.3}$$

for all $a < a_{\mathrm{A}}$, $a' < a_{\mathrm{B}}$, then

$$\sup_{x\geq 0} (f_{\mathrm{A}} * f_{\mathrm{B}})(x)\exp\left(ax^{B}\right) < \infty$$

for all $a < A$, where $B = \min(b_{\mathrm{A}}, b_{\mathrm{B}})$ and

$$A = \begin{cases} a_{\mathrm{A}}a_{\mathrm{B}}\left({a_{\mathrm{A}}}^{1/B} + {a_{\mathrm{B}}}^{1/B}\right)^{-B} & \text{if } b_{\mathrm{A}} = b_{\mathrm{B}} \\ a_{\mathrm{A}} & \text{if } b_{\mathrm{A}} < b_{\mathrm{B}} \\ a_{\mathrm{B}} & \text{if } b_{\mathrm{A}} > b_{\mathrm{B}}. \end{cases}$$

Proof. According to (3.3), f_{A} and f_{B} are bounded and integrable, implying that $f_{\mathrm{A}} * f_{\mathrm{B}}$ is bounded. Therefore the claim holds for $a \leq 0$. Fix $a'' < a < a_{\mathrm{A}}$ and $a''' < a' < a_{\mathrm{B}}$. According to (3.3), there exists $c > 0$ such that, for all $t \in (0,1)$ and $y \geq 0$

$$(f_{\mathrm{A}} * f_{\mathrm{B}})(y) \leq c\int_0^{ty} \exp\left(-a(y-x)^{b_{\mathrm{A}}} - a'x^{b_{\mathrm{B}}}\right)\mathrm{d}x + c\int_{ty}^{y} \exp\left(-a(y-x)^{b_{\mathrm{A}}} - a'x^{b_{\mathrm{B}}}\right)\mathrm{d}x. \tag{3.4}$$

For the first integral one has

$$\int_0^{ty} \exp\left(-a(y-x)^{b_{\mathrm{A}}} - a'x^{b_{\mathrm{B}}}\right)\mathrm{d}x \leq \exp\left(-a(1-t)^{b_{\mathrm{A}}}y^{b_{\mathrm{A}}}\right)\int_0^{\infty} \exp\left(-a'x^{b_{\mathrm{B}}}\right)\mathrm{d}x, \tag{3.5}$$

and because $a''' < a'$, there can be shown to exist $c' > 0$ such that the second integral satisfies

$$\int_{ty}^{y} \exp\left(-a(y-x)^{b_{\mathrm{A}}} - a'x^{b_{\mathrm{B}}}\right)\mathrm{d}x < \int_{ty}^{\infty} \exp\left(-a'x^{b_{\mathrm{B}}}\right)\mathrm{d}x < c'\exp\left(-a'''t^{b_{\mathrm{B}}}y^{b_{\mathrm{B}}}\right). \tag{3.6}$$

By the substitution of (3.5) and (3.6) into (3.4), it follows that there exists $c'' > 0$ such that

$$(f_{\mathrm{A}} * f_{\mathrm{B}})(y) \leq c''\exp\left(-a(1-t)^{b_{\mathrm{A}}}y^{b_{\mathrm{A}}}\right) + c''\exp\left(-a'''t^{b_{\mathrm{B}}}y^{b_{\mathrm{B}}}\right). \tag{3.7}$$

Consider the case $b_{\mathrm{A}} < b_{\mathrm{B}}$. Because convolution commutes, b_{A} and b_{B} are interchangeable and the same reasoning applies to the case $b_B < b_A$ as well. Because $a'' < a$, we can take $t \in (0, 1-(a''/a)^{1/b_A})$. In this case, it is easy to check that

$$\sup_{y\geq 0} \exp\left(-a(1-t)^{b_A} y^{b_A} + a'' y^{b_A}\right) + \exp\left(-a''' t^{b_B} y^{b_B} + a'' y^{b_A}\right) < \infty,$$

and thus, by (3.7), the claim holds for $b_A \neq b_B$. In the case $b_A = b_B = B$, we choose $t = {a_A}^{1/B}/({a_A}^{1/B} + {a_B}^{1/B})$. In this case, $a_A(1-t)^B = a_B t^B = A$ and therefore, for all $a'''' < A$, we can choose $a \in (a''''(1-t)^{-B}, a_A)$ and $a''' \in (a'''' t^{-B}, a_B)$. The substitution of a and a''' in (3.7) yields

$$(f_A * f_B)(y) \leq 2c'' \exp\left(-a'''' y^B\right).$$

□

In order to apply this result to bounding the distribution of ξ_i, we define

$$A_3 \triangleq \begin{cases} A_1 A_2 \left(A_1^{1/B_1}(\alpha\beta + \beta) + A_2^{1/B_1}\right)^{-B_1} & \text{if } B_1 = B_2 \\ A_1 & \text{if } B_1 < B_2. \end{cases}$$

In this case, we have the following proposition, which essentially states the conclusion of Lemma 3.2 in a form more suitable for our purposes.

Proposition 3.3. *For all $a < A_3$, and $i > 1$*

$$\sup_{x\geq 0} \rho_{\xi_i}(x) \exp\left(a x^{B_1}\right) < \infty. \tag{3.8}$$

Proof. By the associativity of the convolution operation, $\rho_{\xi_i} = ((\rho_{\alpha\beta\|V_{i-1}\|} * \rho_{\beta\|V_i\|}) * \rho_{\|W_i\|})$. Moreover, $\rho_{\alpha\beta\|V_i\|}(x) = \rho_{\|V_i\|}(x/\alpha\beta)/\alpha\beta$ and $\rho_{\beta\|V_i\|}(x) = \rho_{\|V_i\|}(x/\beta)/\beta$. Thus, according to (A3) and (A4), one has for all $\epsilon > 0$, and $i > 0$

$$\sup_{x\geq 0} \rho_{\|W_i\|}(x) \exp\left(\left(A_1 - \epsilon\right) x^{B_1}\right) < \infty$$
$$\sup_{x\geq 0} \rho_{\beta\|V_i\|}(x) \exp\left(\left(\frac{A_2}{\beta^{B_2}} - \epsilon\right) x^{B_2}\right) < \infty$$
$$\sup_{x\geq 0} \rho_{\alpha\beta\|V_i\|}(x) \exp\left(\left(\frac{A_2}{(\alpha\beta)^{B_2}} - \epsilon\right) x^{B_2}\right) < \infty$$

and, according to Lemma 3.2, for all $i > 1$ and $\epsilon > 0$

$$\sup_{x\geq 0} (\rho_{\alpha\beta\|V_{i-1}\|} * \rho_{\beta\|V_i\|})(x) \exp\left(\left(A_2/(\alpha\beta + \beta)^{B_2} - \epsilon\right) x^{B_2}\right) < \infty.$$

The claim follows by applying Lemma 3.2 once more to the convolution of $\rho_{\alpha\beta\|V_{i-1}\|} * \rho_{\beta\|V_i\|}$ and $\rho_{\|W_i\|}$. □

By defining

$$A_2^* \triangleq \left(2 + \left(1 + \frac{a_1^{1/b_1}}{A_3^{1/b_1}}\right)^{b_1}\right) a_1(\alpha\beta + \beta)^{b_1},$$

we are ready to prove the uniform convergence of the truncated filter according to the following theorem, which can be regarded as a refinement of the Proposition 3.4 in [56].

Theorem 3.4. *If one of the following conditions*

(i) $b_1 = B_1 < B_2$;

(ii) $b_1 = B_1 = B_2$ *and* $A_2 > A_2^*$,

holds, then there exists c_8, $c_9 > 0$ *such that for all* $i \geq 0$

$$\mathsf{E}\left[\left\|\pi_i - \pi_i^\Delta\right\|_{\mathrm{TV}}\right] \leq c_8 \exp\left(-c_9 \Delta^{B_1}\right). \tag{3.9}$$

Proof. Let us write

$$J_i(\Delta) = \min\left(1, \frac{\left\|\pi_{i,i}^\Delta - \pi_{i,i-1}^\Delta\right\|_{\mathrm{TV}}}{\varepsilon_{i+1}(\Delta)}\right).$$

Because $J_i(\Delta)$ is $\mathscr{F}_{i+1}$-measurable, it follows from Proposition 2.7, Lemma 2.9, and Lemma 2.12 that

$$\begin{aligned}
\mathsf{E}\left[\left\|\pi_i - \pi_i^\Delta\right\|_{\mathrm{TV}}\right] &\leq \sum_{j=1}^{i} \mathsf{E}\left[\mathsf{E}\left[\prod_{n=j+2}^{i}(1-\varepsilon_n(\Delta)) \,\middle|\, \mathscr{F}_{j+1}\right] J_j(\Delta)\right] \\
&= \sum_{j=1}^{i} \mathsf{E}\left[\mathsf{E}\left[\prod_{n=1}^{i-j-1}(1-\varepsilon_{n+j+1}(\Delta)) \,\middle|\, \mathscr{F}_{j+1}\right] J_j(\Delta)\right] \\
&\leq \sum_{j=1}^{i}(1-\tilde{\varepsilon}(\Delta))^{(i-j-2)^+} \mathsf{E}\left[J_j(\Delta)\right],
\end{aligned} \tag{3.10}$$

where $(\cdot)^+ \triangleq \max(0,\cdot)$. Next we derive an upper bound for $\mathsf{E}\left[J_j(\Delta)\right]$ which is independent of j and therefore can be brought outside the summation. For the remaining sum, it follows by the convergence of geometric series that

$$\sum_{j=1}^{i}(1-\tilde{\varepsilon}(\Delta))^{(i-j-2)^+} \leq 2 + \frac{1}{\tilde{\varepsilon}(\Delta)}. \tag{3.11}$$

It can be shown that for all $\epsilon > 0$ there exists $c = c(\epsilon) > 0$ such that for all $\Delta > \Delta_0$ and

$i > 0$ one has

$$\eta_\epsilon(\Delta) \triangleq c\exp\Big((-A_2+\epsilon)(\Delta-\gamma)^{B_2}\Big) \geq \int_{\|y\|>\Delta-\gamma} \rho_{V_i}(y)\mathrm{d}y.$$

Moreover, it is elementary to show that for all $u, v \geq 0$,

$$\min(1, uv) \leq \min(1, u) + \min(1, v). \tag{3.12}$$

Therefore, by applying (3.12) to $\mathsf{E}\big[J_i(\Delta)\big]$, one has

$$\mathsf{E}\big[J_i(\Delta)\big] \leq \mathsf{E}\left[\min\left(1, \frac{\left\|\pi^\Delta_{i,i} - \pi^\Delta_{i,i-1}\right\|_{\mathrm{TV}}}{\eta_\epsilon(\Delta)^q}\right) + \min\left(1, \frac{\eta_\epsilon(\Delta)^q}{\varepsilon_{i+1}(\Delta)}\right)\right], \tag{3.13}$$

where $q \in \mathbb{R}$. According to Lemma 2.3(i), $\left\|\pi^\Delta_{i,i} - \pi^\Delta_{i,i-1}\right\|_{\mathrm{TV}} = \pi_i(\complement C_i(\Delta))$, and therefore

$$\mathsf{E}\left[\min\left(1, \frac{\left\|\pi^\Delta_{i,i} - \pi^\Delta_{i,i-1}\right\|_{\mathrm{TV}}}{\eta_\epsilon(\Delta)^q}\right)\right] \leq \mathsf{E}\left[\frac{\left\|\pi^\Delta_{i,i} - \pi^\Delta_{i,i-1}\right\|_{\mathrm{TV}}}{\eta_\epsilon(\Delta)^q}\right] = \frac{\mathsf{E}\big[\pi_i(\complement C_i(\Delta))\big]}{\eta_\epsilon(\Delta)^q}. \tag{3.14}$$

Moreover, according to (2.4), for all $i > 0$

$$\begin{aligned}
\mathsf{E}\big[\pi_i(\complement C_i(\Delta))\big] &= \mathsf{E}\left[\mathsf{E}\left[\frac{\int_{\complement C_i(\Delta)} g_{i,Y_i}\,\mathrm{d}\pi_{i-1}K_i}{\int g_{i,Y_i}\,\mathrm{d}\pi_{i-1}K_i} \,\middle|\, \mathscr{Y}_{i-1}\right]\right] \\
&= \mathsf{E}\left[\int \left[\frac{\int_{\complement C_i(\Delta)} g_{i,y}\,\mathrm{d}\pi_{i-1}K_i}{\int g_{i,y}\,\mathrm{d}\pi_{i-1}K_i}\int g_{i,y}\,\mathrm{d}\pi_{i-1}K_i\right]\mathrm{d}y\right] \\
&= \mathsf{E}\left[\int \left[\int_{\complement C_i(\Delta)} g_{i,y}\,\mathrm{d}\pi_{i-1}K_i\right]\mathrm{d}y\right] \\
&= \mathsf{E}\left[\int \left[\int g_{i,y}(x)1_{\complement D_i(\Delta)}(x,y)\,\mathrm{d}y\right]\pi_{i-1}K_i(\mathrm{d}x)\right],
\end{aligned} \tag{3.15}$$

where $D_i(\Delta) \triangleq \{(x,y) \in \mathbb{R}^{d_\mathrm{s}} \times \mathbb{R}^{d_\mathrm{m}} \mid \|y - \tilde{h}_i(x)\| \leq \Delta\}$, and the last equality follows from Fubini's theorem. By the change of variable, the inner integral satisfies

$$\int g_{i,y}(x)1_{\complement D_i(\Delta)}(x,y)\mathrm{d}y = \int 1_{\complement D_i(\Delta)}(x, z+h_i(x))\rho_{V_i}(z)\mathrm{d}z \leq \int_{\|z\|>\Delta-\gamma} \rho_{V_i}(z)\mathrm{d}z \leq \eta_\epsilon(\Delta), \tag{3.16}$$

where the first inequality follows from the observation that

$$\|z\| + \gamma \geq \|z\| + \left\|\bar{h}_i(x)\right\| \geq \left\|z + \bar{h}_i(x)\right\|,$$

and therefore

$$1_{\complement D_i(\Delta)}(x, z + h(x)) = 1_{\{(x,z) \mid \|z + \bar{h}_i(x)\| > \Delta\}}(x, z) \leq 1_{\{(x,z) \mid \|z\| > \Delta - \gamma\}}(x, z).$$

By putting (3.14), (3.15), and (3.16) together one has

$$\mathsf{E}\left[\min\left(1, \frac{\left\|\pi_{i,i}^{\Delta} - \pi_{i,i-1}^{\Delta}\right\|_{\mathrm{TV}}}{\eta_\epsilon(\Delta)^q}\right)\right] \leq \eta_\epsilon(\Delta)^{1-q} = c^{1-q} \exp\left((1-q)(-A_2 + \epsilon)(\Delta - \gamma)^{B_2}\right). \tag{3.17}$$

Let us then consider the second integral in (3.13). According to Proposition 3.3 for all $i > 0$

$$\begin{aligned}
\mathsf{E}\left[\min\left(1, \frac{\eta_\epsilon(\Delta)^q}{\varepsilon_{i+1}(\Delta)}\right)\right] &= \int_0^\infty \min\left(1, \frac{\eta_\epsilon(\Delta)^q}{\varepsilon(\Delta, x)}\right) \rho_{\xi_{i+1}}(x) \mathrm{d}x \\
&\leq c' \int_0^{\theta\Delta} \frac{\eta_\epsilon(\Delta)^q}{\varepsilon(\Delta, x)} \exp\left((-A_3 + \epsilon) x^{B_1}\right) \mathrm{d}x \\
&\quad + c' \int_{\theta\Delta}^\infty \exp\left((-A_3 + \epsilon) x^{B_1}\right) \mathrm{d}x \\
&\leq c' \int_0^\infty \exp\left((-A_3 + \epsilon) x^{B_1}\right) \mathrm{d}x \, \frac{\eta_\epsilon(\Delta)^q}{\varepsilon(\Delta, \theta\Delta)} \\
&\quad + c'' \exp\left((-A_3 + 2\epsilon)(\theta\Delta)^{B_1}\right),
\end{aligned} \tag{3.18}$$

where the second inequality follows from the fact that $\eta_\epsilon(\Delta)^q / \varepsilon(\Delta, \cdot)$ is nondecreasing. Therefore, by combining (3.10), (3.11), (3.17), and (3.18), one has

$$\mathsf{E}\left[\left\|\pi_i - \pi_i^{\Delta}\right\|_{\mathrm{TV}}\right] \leq \left(2 + \frac{1}{\tilde{\varepsilon}(\Delta)}\right)\left(\eta_\epsilon(\Delta)^{1-q} + \frac{c''' \eta_\epsilon(\Delta)^q}{\varepsilon(\Delta, \theta\Delta)} + c'' \exp\left((-A_3 + 2\epsilon)(\theta\Delta)^{B_1}\right)\right).$$

In order to prove the claim, it then remains to show that each of the terms

$$\begin{aligned}
\frac{\eta_\epsilon(\Delta)^{1-q}}{\tilde{\varepsilon}(\Delta)} &\propto \exp\left((1-q)(-A_2 + \epsilon)(\Delta - \gamma)^{B_2} + a_1(L(\Delta) + r)^{b_1}\right) \\
\frac{\eta_\epsilon(\Delta)^q}{\tilde{\varepsilon}(\Delta)\varepsilon(\Delta, \theta\Delta)} &\propto \exp\left(q(-A_2 + \epsilon)(\Delta - \gamma)^{B_2} + a_1(L(\Delta) + r)^{b_1} + a_1(L(\Delta) + \theta\Delta)^{b_1}\right) \\
\frac{\exp\left((-A_3 + 2\epsilon)(\theta\Delta)^{B_1}\right)}{\tilde{\varepsilon}(\Delta)} &\propto \exp\left((-A_3 + 2\epsilon)(\theta\Delta)^{B_1} + a_1(L(\Delta) + r)^{b_1}\right),
\end{aligned}$$

converges to zero at appropriate rate, as $\Delta \to \infty$. In the case (i) this holds if θ is sufficiently large, and ϵ is sufficiently close to zero. In the case (ii), ϵ must be sufficiently close to zero and (q, θ) must be a solution of

$$\begin{cases} (1-q)A_2 & > & a_1(\alpha\beta+\beta)^{b_1} \\ qA_2 & > & a_1(\alpha\beta+\beta+\theta)^{b_1}+a_1(\alpha\beta+\beta)^{b_1} \\ A_3\theta^{B_1} & > & a_1(\alpha\beta+\beta)^{b_1}, \end{cases}$$

One can check that a solution to this system of inequalities exists if $A_2 > A_2^*$. □

3.2 Uniform convergence theorem

In this section, we consider an approximation $\tilde{\pi}^\Delta$ of π^Δ and show that under certain conditions this approximation is uniform and that the error of $\tilde{\pi}^\Delta$ converges to zero as $\Delta \to \infty$. Therefore, provided that the conditions of Theorem 3.4 are satisfied, $\tilde{\pi}^\Delta$ is a uniformly convergent approximation of π as well. For this purpose, we define $\mathscr{Y} \triangleq \bigcup_{i\geq 0} \mathscr{Y}_i$ and $\mathscr{H} \triangleq (\mathscr{H}_i)_{i\geq 0}$, such that $\mathscr{H}$ is a nondecreasing sequence of sub-σ-fields $\mathscr{H}_i \subset \mathscr{F}$ satisfying $\mathscr{H}_0 = \mathscr{Y}$. Moreover, we let K_i^Δ denote the restriction of K_i to the set $C_i(\Delta)$, i.e. for all $x \in \mathbb{R}^{d_s}$ and $A \in \mathscr{B}(\mathbb{R}^{d_s})$, one has $K_i^\Delta(x,A) = K_i(x, C_i(\Delta) \cap A)$. Note that $K_i^\Delta(x,\cdot)$ is a finite measure but not necessarily a probability measure on $\mathscr{B}(\mathbb{R}^{d_s})$.

The following assumptions are made about $\tilde{\pi}^\Delta$:

(A5) For all $i \geq 0$, $\tilde{\pi}_i^\Delta$ is $\mathscr{H}_i$-measurable, $\tilde{\pi}_0^\Delta = \pi_0$ and

$$\tilde{\pi}_i^\Delta = g_i^\Delta \cdot \nu_i^\Delta, \tag{3.19}$$

where $\nu_i^\Delta \in M_F(\mathscr{B}(\mathbb{R}^{d_s}))$ and $0 < \nu_i^\Delta(g_i^\Delta) < \infty$, P-a.s.

(A6) There exist a_4, $M_4 > 0$ such that for all $\Delta > \Delta_0$ and $i > 0$

$$\sup_{\|\varphi\|_\infty \leq 1} \mathsf{E}\left[\left|\tilde{\pi}_{i-1}^\Delta K_i^\Delta(\varphi) - \nu_i^\Delta(\varphi)\right| \,\Big|\, \mathscr{H}_{i-1}\right] \leq M_4 \exp\left(-a_4\Delta^{b_2}\right).$$

Moreover, the following additional assumption about the filter framework is considered:

(A4') For all $x \in \mathbb{R}^{d_s}$,

$$\rho_{X_0/\beta}(x) \leq M_2 \exp\left(-A_2 \|x\|^{B_2}\right),$$

where $\rho_{X_0/\beta}$ is the density of the random variable X_0/β with respect to λ_{d_s}.

Note that (A4') is only related to the filter framework and therefore it does not impose any additional restrictions on the approximating algorithm. Moreover, (A4') is not crucial for the proof of convergence but under this assumption, explicit rates for convergence can be obtained.

In the proof of the main theorem on the uniform convergence, the following general result is needed [see also 56, Lemma 5.2]:

Lemma 3.5. *Suppose that K is a random transition probability in $\mathbb{R}^{d_s} \times \mathscr{B}(\mathbb{R}^{d_s})$ and that μ and ν are random probability measures on $\mathscr{B}(\mathbb{R}^{d_s})$ with a random support $C \in \mathscr{B}(\mathbb{R}^{d_s})$ such that $\alpha_C(K)$ is $\mathscr{G}$-measurable where $\mathscr{G} \subset \mathscr{F}$. Let ψ and θ be bounded, nonnegative random functions such that $\mu(\psi)$ and $\nu(\theta)$ are positive random variables. Then,*

$$\text{(i)}\quad \frac{1}{2}\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\Big[\big|\mu K(\varphi) - \nu K(\varphi)\big| \,\Big|\, \mathscr{G}\Big] \le (1-\alpha_C(K)) \sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\Big[\big|\mu\varphi - \nu\varphi\big| \,\Big|\, \mathscr{G}\Big];$$

$$\text{(ii)}\quad \frac{1}{2}\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\Big[\big|(\psi\cdot\mu)(\varphi) - (\theta\cdot\nu)(\varphi)\big| \,\Big|\, \mathscr{G}\Big] \le \sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\left[\left.\frac{\big|\mu(\psi\varphi) - \nu(\theta\varphi)\big|}{\mu\psi}\,\right|\, \mathscr{G}\right].$$

Proof. To prove (i), it is observed that for all $x \in C$

$$\begin{aligned}\big|\mu K(\varphi) - \nu K(\varphi)\big| &= \big|\mu(K\varphi - K\varphi(x)) - \nu(K\varphi - K\varphi(x))\big| \\ &\le \left|\frac{\mu(K\varphi - K\varphi(x))}{\big\|1_C(K\varphi - K\varphi(x))\big\|_\infty} - \frac{\nu(K\varphi - K\varphi(x))}{\big\|1_C(K\varphi - K\varphi(x))\big\|_\infty}\right| \\ &\quad \times \sup_{\|\bar\varphi\|_\infty \le 1} \big\|1_C(K\bar\varphi - K\bar\varphi(x))\big\|_\infty,\end{aligned}$$

and

$$\begin{aligned}\sup_{\|\bar\varphi\|_\infty \le 1} \big\|1_C(K\bar\varphi - K\bar\varphi(x))\big\|_\infty &\le \sup_{\substack{x,z\in C\\ \|\bar\varphi\|_\infty \le 1}} \big|K\bar\varphi(z) - K\bar\varphi(x)\big| \\ &= 2\sup_{\substack{x,z\in C\\ A\in\mathscr{B}(\mathbb{R}^{d_s})}} |K(z,A) - K(x,A)| = 2(1-\alpha_C(K)).\end{aligned}$$

Because $\alpha_C(K)$ is $\mathscr{G}$-measurable, the claim follows by taking conditional expectations. To prove (ii) it is observed that [see e.g., 15, 47],

$$\big|(\psi\cdot\mu)(\varphi) - (\theta\cdot\nu)(\varphi)\big| \le \frac{\big|\mu(\psi\varphi) - \nu(\theta\varphi)\big|}{\mu\psi} + \|\varphi\|_\infty \frac{\big|\mu\psi - \nu\theta\big|}{\mu\psi}. \tag{3.20}$$

Because

$$\mathsf{E}\left[\frac{|\mu\psi-\nu\theta|}{\mu\psi}\,\middle|\,\mathscr{G}\right]\le\sup_{\|\varphi\|_\infty\le1}\mathsf{E}\left[\frac{|\mu(\psi\varphi)-\nu(\theta\varphi)|}{\mu\psi}\,\middle|\,\mathscr{G}\right],$$

the claim follows by taking conditional expectations in (3.20) and by using the fact that $\|\varphi\|_\infty\le 1$. □

By defining

$$a_4^*\triangleq 2\left(1+\left(1+\frac{a_1^{1/b_1}}{A_3^{1/b_1}}\right)^{b_1}\right)a_1(\alpha\beta+\beta)^{b_1}+a_2,$$

we are ready to state the main result regarding the uniform convergence.

Theorem 3.6. *If one of the following conditions holds:*

(i) $b_1=B_1<B_2\le b_2$ *and* $a_2<a_4$*;*

(ii) $b_1=B_1=B_2<b_2$, $a_2<a_4$, *and* $A_2^*<A_2$*;*

(iii) $b_1=B_1=B_2=b_2$, $a_4^*<a_4$, *and* $A_2^*<A_2$*;*

then

$$\lim_{\Delta\to\infty}\mathsf{E}\left[\sup_{\|\varphi\|_\infty\le1}\mathsf{E}\left[\left|\pi_i\varphi-\tilde{\pi}_i^\Delta\varphi\right|\,\middle|\,\mathscr{Y}_i\right]\right]=0. \tag{3.21}$$

If in addition (A4') holds, then there exists c_{10}, $c_{11}>0$ *such that*

$$\mathsf{E}\left[\sup_{\|\varphi\|_\infty\le1}\mathsf{E}\left[\left|\pi_i\varphi-\tilde{\pi}_i^\Delta\varphi\right|\,\middle|\,\mathscr{Y}_i\right]\right]\le c_{10}\exp\left(-c_{11}\Delta^{B_1}\right). \tag{3.22}$$

Proof. The proof is based on similar principles as the proof of the filter stability in Chapter 2. Therefore, following the definition of $\pi_{i,j}^\Delta$ in (2.15), we define for all $i\ge j\ge 0$,

$$\tilde{\pi}_{i,j}^\Delta\triangleq Q_{j+1,i}^\Delta(\tilde{\pi}_j^\Delta). \tag{3.23}$$

Accordingly, $\tilde{\pi}_{i,i}^\Delta=\tilde{\pi}_i^\Delta$ and $\tilde{\pi}_{i,0}^\Delta=\pi_i^\Delta$. By the triangle inequality,

$$\begin{aligned}\mathsf{E}\left[\sup_{\|\varphi\|_\infty\le1}\mathsf{E}\left[\left|\pi_i\varphi-\tilde{\pi}_i^\Delta\varphi\right|\,\middle|\,\mathscr{Y}_i\right]\right]&\le\mathsf{E}\left[\sup_{\|\varphi\|_\infty\le1}\mathsf{E}\left[\left|\pi_i\varphi-\pi_i^\Delta\varphi\right|\,\middle|\,\mathscr{Y}_i\right]\right]\\&\quad+\mathsf{E}\left[\sup_{\|\varphi\|_\infty\le1}\mathsf{E}\left[\left|\tilde{\pi}_i^\Delta\varphi-\pi_i^\Delta\varphi\right|\,\middle|\,\mathscr{Y}_i\right]\right].\end{aligned} \tag{3.24}$$

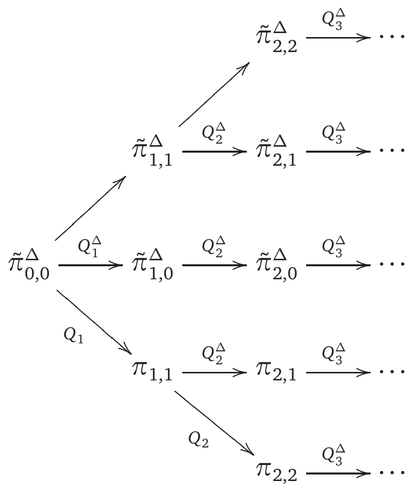

Figure 3.1: The exact filter and its approximations.

See also Figure 3.1 for illustration. For the first expectation on the right hand side of (3.24), one has

$$\mathsf{E}\left[\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\left[\left|\pi_i\varphi - \pi_i^\Delta\varphi\right| \,\middle|\, \mathscr{Y}_i\right]\right] \le \mathsf{E}\left[\sup_{\|\varphi\|_\infty \le 1} \left|\pi_i\varphi - \pi_i^\Delta\varphi\right|\right] = 2\mathsf{E}\left[\left\|\pi_i - \tilde{\pi}_i^\Delta\right\|_{\mathrm{TV}}\right]. \tag{3.25}$$

According to Theorem 3.4, $\mathsf{E}\left[\left\|\pi_i - \pi_i^\Delta\right\|_{\mathrm{TV}}\right]$ is uniformly convergent to zero as $\Delta \to \infty$ for all cases (i), (ii), and (iii). Therefore it suffices to consider only the second term on the right hand side of (3.24). Because for all $i, j > 0$, $\mathscr{Y}_i \subset \mathscr{H}_j$, one has

$$\begin{aligned}
\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\left[\left|\tilde{\pi}_i^\Delta\varphi - \pi_i^\Delta\varphi\right| \,\middle|\, \mathscr{Y}_i\right] &\le \sum_{j=1}^{i} \sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\left[\mathsf{E}\left[\left|\tilde{\pi}_{i,j}^\Delta\varphi - \tilde{\pi}_{i,j-1}^\Delta\varphi\right| \,\middle|\, \mathscr{H}_{j-1}\right] \,\middle|\, \mathscr{Y}_i\right] \\
&\le \sum_{j=1}^{i} \sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\left[\left|\tilde{\pi}_{i,j}^\Delta\varphi - \tilde{\pi}_{i,j-1}^\Delta\varphi\right| \,\middle|\, \mathscr{H}_{j-1}\right].
\end{aligned} \tag{3.26}$$

According to Lemma 2.4 and (3.23),

$$\tilde{\pi}_{i,j}^\Delta = Q_{j+1,i}^\Delta(\tilde{\pi}_{j,j}^\Delta) = (\psi_{j+1,i}^\Delta \cdot \tilde{\pi}_{j,j}^\Delta) S_{j+1,i}^\Delta \cdots S_{i,i}^\Delta$$

$$\tilde{\pi}_{i,j-1}^\Delta = Q_{j+1,i}^\Delta(\tilde{\pi}_{j,j-1}^\Delta) = (\psi_{j+1,i}^\Delta \cdot \tilde{\pi}_{j,j-1}^\Delta) S_{j+1,i}^\Delta \cdots S_{i,i}^\Delta$$

and therefore according to Lemma 3.5(i)

$$\sup_{\|\varphi\|_\infty\le 1} \mathsf{E}\left[\left|\tilde{\pi}^\Delta_{i,j}\varphi - \tilde{\pi}^\Delta_{i,j-1}\varphi\right| \,\Big|\, \mathscr{H}_{j-1}\right] \le 4\left(1-\alpha_{C_j(\Delta)}(S^\Delta_{j+1,i}\cdots S^\Delta_{i,i})\right) J_{j,i}(\Delta), \tag{3.27}$$

where

$$J_{j,i}(\Delta) \triangleq \frac{1}{2}\sup_{\|\varphi\|_\infty\le 1} \mathsf{E}\left[\left|\left(\psi^\Delta_{j+1,i}\cdot\tilde{\pi}^\Delta_{j,j}\right)(\varphi) - \left(\psi^\Delta_{j+1,i}\cdot\tilde{\pi}^\Delta_{j,j-1}\right)(\varphi)\right| \,\Big|\, \mathscr{H}_{j-1}\right]. \tag{3.28}$$

It is then observed that according to (A5)

$$\psi^\Delta_{j+1,i}\cdot\tilde{\pi}^\Delta_{j,j} = \psi^\Delta_{j+1,i}\cdot(g^\Delta_j\cdot\nu^\Delta_j) = g^\Delta_j\psi^\Delta_{j+1,i}\cdot\nu^\Delta_j,$$
$$\psi^\Delta_{j+1,i}\cdot\tilde{\pi}^\Delta_{j,j-1} = \psi^\Delta_{j+1,i}\cdot(g^\Delta_j\cdot\tilde{\pi}^\Delta_{j-1}K_j) = g^\Delta_j\psi^\Delta_{j+1,i}\cdot\tilde{\pi}^\Delta_{j-1}K_j.$$

Thus, according to Lemma 3.5(ii) and the fact that $\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i})$ is $\mathscr{H}_{j-1}$-measurable, we have

$$\begin{aligned} J_{j,i}(\Delta) &= \frac{1}{2}\sup_{\|\varphi\|_\infty\le 1} \mathsf{E}\left[\left|\left(g^\Delta_j\psi^\Delta_{j+1,i}\cdot\nu^\Delta_j\right)(\varphi) - \left(g^\Delta_j\psi^\Delta_{j+1,i}\cdot\tilde{\pi}^\Delta_{j-1}K_j\right)(\varphi)\right| \,\Big|\, \mathscr{H}_{j-1}\right] \\ &\le \frac{1}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i})}\sup_{\|\varphi\|_\infty\le 1} \mathsf{E}\left[\left|\nu^\Delta_j(g^\Delta_j\psi^\Delta_{j+1,i}\varphi) - \tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i}\varphi)\right| \,\Big|\, \mathscr{H}_{j-1}\right] \\ &= \frac{\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i})}\sup_{\|\varphi\|_\infty\le 1} \mathsf{E}\left[\left|\frac{\nu^\Delta_j(g^\Delta_j\psi^\Delta_{j+1,i}\varphi)}{\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty} - \frac{\tilde{\pi}^\Delta_{j-1}K^\Delta_j(g^\Delta_j\psi^\Delta_{j+1,i}\varphi)}{\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty}\right| \,\Bigg|\, \mathscr{H}_{j-1}\right], \end{aligned}$$

where the last equality follows from the fact that $\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty$ is $\mathscr{H}_{j-1}$-measurable. Also, observe that K_j can be replaced by K^Δ_j because g^Δ_j is supported by $C_j(\Delta)$. Therefore, it follows from (A6) that

$$J_{j,i}(\Delta) \le \frac{\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i})} M_4 \exp\left(-a_4\Delta^{b_2}\right). \tag{3.29}$$

Moreover,

$$\frac{\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i})} = \frac{\left\|g^\Delta_j\psi^\Delta_{j+1,i}\right\|_\infty}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j)}\,\frac{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j)}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j\psi^\Delta_{j+1,i})} \le \frac{\left\|g^\Delta_j\right\|_\infty}{\tilde{\pi}^\Delta_{j-1}K_j(g^\Delta_j)}\,\frac{\left\|\psi^\Delta_{j+1,i}\right\|_\infty}{\tilde{\pi}^\Delta_{j,j-1}(\psi^\Delta_{j+1,i})}. \tag{3.30}$$

In order to bound the second product term in (3.30) it is observed that similarly as in the proof of Proposition 2.7, for all $j > 0$

$$\begin{aligned}\frac{\psi^{\Delta}_{j+1,i}(x)}{\tilde{\pi}^{\Delta}_{j,j-1}(\psi^{\Delta}_{j+1,i})} &= \frac{K_{j+1}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i})(x)}{\tilde{\pi}^{\Delta}_{j,j-1}(1_{C_j(\Delta)}K_{j+1}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i}))}\\ &\leq \frac{\left\|k_{j+1}\right\|_{\infty}}{\inf_{\substack{x\in C_j(\Delta)\\ y\in C_{j+1}(\Delta)}} k_{j+1}(x,y)} \frac{\lambda_{d_s}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i})}{\tilde{\pi}^{\Delta}_{j,j-1}(\lambda_{d_s}(g^{\Delta}_{j+1}\psi^{\Delta}_{j+2,i}))} \leq \frac{1}{\varepsilon_{j+1}(\Delta)},\end{aligned} \tag{3.31}$$

where the last inequality follows from Lemma 2.9. In order to bound the first product term in (3.30) it is observed that for all $j > 0$,

$$\begin{aligned}\tilde{\pi}^{\Delta}_{j-1}K_j(g^{\Delta}_j) &\geq \int_{C_{j-1}(\Delta)}\left[\int_{C_j(\Delta)} g_j(y)k_j(x,y)\mathrm{d}y\right]\tilde{\pi}^{\Delta}_{j-1}(\mathrm{d}x)\\ &\geq \lambda_{d_s}(C_j(\Delta))\tilde{\pi}^{\Delta}_{j-1}(C_{j-1}(\Delta))\inf_{x\in C_j(\Delta)} g_j(x)\inf_{\substack{y\in C_j(\Delta)\\ x\in C_{j-1}(\Delta)}} k_j(x,y).\end{aligned} \tag{3.32}$$

Note that the first inequality can be replaced by equality for all $j > 1$ but because $\tilde{\pi}^{\Delta}_0 = \pi_0$, the equality does not hold for $j = 1$. For all $x \in B_{d_s}(\tilde{h}_j^{-1}(Y_j), \Delta_0/\beta_0)$, one has

$$\|Y_j - \tilde{h}_j(x)\| = \|\tilde{h}_j(\tilde{h}_j^{-1}(Y_j)) - \tilde{h}_j(x)\| \leq \beta_0\|\tilde{h}_j^{-1}(Y_j) - x\| \leq \Delta_0,$$

that is, $B_{d_s}(\tilde{h}_j^{-1}(Y_j), \Delta_0/\beta_0) \subset C_j(\Delta_0)$ and thus for all $\Delta > \Delta_0$, $B_{d_s}(\tilde{h}_j^{-1}(Y_j), \Delta_0/\beta_0) \subset C_j(\Delta)$ which implies $\lambda_{d_s}(C_j(\Delta)) \geq \lambda_{d_s}(B_{d_s}(\tilde{h}_j^{-1}(Y_j), \Delta_0/\beta_0)) = \tilde{V}_{d_s}(\Delta_0/\beta_0)$, where $\tilde{V}_{d_s}(\Delta_0/\beta_0)$ denotes the volume of a d_s dimensional ball of radius Δ_0/β_0. According to the assumptions (A2) and (A4)

$$\begin{aligned}\inf_{x\in C_j(\Delta)} g_j(x) &\geq \inf_{x\in C_j(\Delta)} m_2\exp\left(-a_2\left\|Y_j - h_j(x)\right\|^{b_2}\right)\\ &\geq \inf_{x\in C_j(\Delta)} m_2\exp\left(-a_2\left(\|Y_j - \tilde{h}_j(x)\| + \|\bar{h}_j(x)\|\right)^{b_2}\right)\\ &\geq m_2\exp\left(-a_2(\Delta+\gamma)^{b_2}\right).\end{aligned}$$

Because $\tilde{\pi}^{\Delta}_j(C_j(\Delta)) = 1$, $j > 0$, one has $\tilde{\pi}^{\Delta}_{j-1}(C_{j-1}(\Delta)) \geq \pi_0(C_0(\Delta_0))$ and therefore, according to (3.32)

$$\tilde{\pi}^{\Delta}_{j-1}K_j(g^{\Delta}_j) \geq \tilde{V}_{d_s}(\Delta_0/\beta_0)\pi_0(C_0(\Delta_0))m_1m_2\exp\left(-a_2(\Delta+\gamma)^{b_2}\right)\varepsilon_j(\Delta), \tag{3.33}$$

where (2.17) and Lemma 2.9 have also been used. From (3.28) it follows that $J_{j,i}(\Delta) \leq 1$

and therefore by putting (3.29), (3.30), (3.31), and (3.33) together one has for all $j > 0$

$$J_j(\Delta) \triangleq \min\left(1, \frac{\Gamma(\Delta)}{\varepsilon_j(\Delta)\varepsilon_{j+1}(\Delta)}\right) \geq J_{j,i}(\Delta). \tag{3.34}$$

where

$$\Gamma(\Delta) \triangleq c\exp\Big(a_2(\Delta+\gamma)^{b_2} - a_4\Delta^{b_2}\Big),$$

with $c \triangleq M_2M_4/\tilde{V}_{d_s}(\Delta_0/\beta_0)\pi_0(C_0(\Delta_0))m_1m_2$. By induction, it follows that

$$1-\alpha_{C_j(\Delta)}(S^{\Delta}_{j+1,i}\cdots S^{\Delta}_{i,i}) \leq \prod_{n=j+1}^{i}\Big(1-\alpha_{C_{n-1}(\Delta)}\left(S^{\Delta}_{n,i}\right)\Big) \leq \prod_{n=j+2}^{i}(1-\varepsilon_n(\Delta)),$$

where the second inequality follows from Lemma 2.6 and Lemma 2.9. Therefore, and also because $J_j(\Delta)$ is $\mathscr{F}_{j+1}$-measurable it follows from (3.26), (3.27), and (3.34) that

$$\begin{aligned}\mathsf{E}\left[\sup_{\|\varphi\|_\infty\leq 1}\mathsf{E}\Big[\big|\tilde{\pi}^{\Delta}_i\varphi - \pi^{\Delta}_i\varphi\big|\,\Big|\,\mathscr{Y}_i\Big]\right] &\leq 4\sum_{j=1}^{i}\mathsf{E}\left[\mathsf{E}\left[\prod_{n=j+2}^{i}(1-\varepsilon_n(\Delta))\,\middle|\,\mathscr{F}_{j+1}\right]J_j(\Delta)\right]\\ &\leq 4\sum_{j=1}^{i}(1-\tilde{\varepsilon}(\Delta))^{(i-j-2)^+}\mathsf{E}\Big[J_j(\Delta)\Big],\end{aligned} \tag{3.35}$$

where the second inequality follows from Lemma 2.12 similarly as in (3.10). According to (3.11)

$$\sum_{j=1}^{i}(1-\tilde{\varepsilon}(\Delta))^{(i-j-2)^+}\mathsf{E}\Big[J_j(\Delta)\Big] \leq \mathsf{E}\big[J_1(\Delta)\big] + \left(2+\frac{1}{\tilde{\varepsilon}(\Delta)}\right)\sup_{j>1}\mathsf{E}\Big[J_j(\Delta)\Big], \tag{3.36}$$

and by applying (3.12)

$$\mathsf{E}\Big[J_j(\Delta)\Big] \leq \mathsf{E}\left[\min\left(1, \frac{\sqrt{\Gamma(\Delta)}}{\varepsilon_j(\Delta)}\right)\right] + \mathsf{E}\left[\min\left(1, \frac{\sqrt{\Gamma(\Delta)}}{\varepsilon_{j+1}(\Delta)}\right)\right]. \tag{3.37}$$

Similarly as in (3.18), for all $\theta > 0$

$$\mathsf{E}\left[\min\left(1, \frac{\sqrt{\Gamma(\Delta)}}{\varepsilon_j(\Delta)}\right)\right] \leq \frac{\sqrt{\Gamma(\Delta)}}{\varepsilon(\Delta,\theta\Delta)} + \int_{\theta\Delta}^{\infty}\rho_{\xi_j}(x)\mathrm{d}x, \tag{3.38}$$

where, according to Proposition 3.3, the integral can be bounded for all $j > 1$ by

$$\int_{\theta\Delta}^{\infty} \rho_{\xi_j}(x)\mathrm{d}x \le c' \exp\Big((-A_3 + 2\epsilon)(\theta\Delta)^{B_1}\Big), \tag{3.39}$$

where $\epsilon > 0$. Thus, by combining (3.35), (3.36), and (3.39) it is observed that for all θ, $\epsilon > 0$, there exists $c'' > 0$ such that

$$\mathsf{E}\left[\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\Big[\big|\tilde{\pi}_i^\Delta \varphi - \pi_i^\Delta \varphi\big| \,\Big|\, \mathscr{Y}_i\Big]\right] \le$$
$$c''\left(\left(1 + \frac{1}{\tilde{\varepsilon}(\Delta)}\right)\left(\frac{\sqrt{\Gamma(\Delta)}}{\varepsilon(\Delta, \theta\Delta)} + \exp\Big((-A_3 + 2\epsilon)(\theta\Delta)^{B_1}\Big)\right) + \int_{\theta\Delta}^{\infty} \rho_{\xi_1}(x)\mathrm{d}x\right). \tag{3.40}$$

In order to prove (3.21) it suffices to consider the convergence of the terms

$$\frac{\sqrt{\Gamma(\Delta)}}{\tilde{\varepsilon}(\Delta)\varepsilon(\Delta, \theta\Delta)} \propto \exp\left(\frac{a_2}{2}(\Delta + \gamma)^{b_2} - \frac{a_4}{2}\Delta^{b_2} + a_1(L(\Delta) + r)^{b_1} + a_1(L(\Delta) + \theta\Delta)^{b_1}\right)$$
$$\frac{\exp\big((-A_3 + 2\epsilon)(\theta\Delta)^{B_1}\big)}{\tilde{\varepsilon}(\Delta)} \propto \exp\Big((-A_3 + 2\epsilon)(\theta\Delta)^{B_1} + a_1(L(\Delta) + r)^{b_1}\Big).$$

If (i) or (ii) is satisfied, θ is sufficiently large, and ϵ is sufficiently close to zero, then these terms converge to zero as $\Delta \to \infty$. In the case (iii), this holds if ϵ is sufficiently close to zero, and θ is a solution of

$$\begin{cases} a_4/2 & > & a_2/2 + a_1(\alpha\beta + \beta + \theta)^{b_1} + a_1(\alpha\beta + \beta)^{b_1} \\ A_3\theta^{b_1} & > & a_1(\alpha\beta + \beta)^{b_1}. \end{cases}$$

A solution to this system of inequalities exists if $a_4 > a_4^*$, which completes the proof of (3.21). If (A4') holds, then the conclusion of Proposition 3.3 applies to ξ_1 as well and therefore (3.39) holds also for $j = 1$. This implies that integral on the right hand side of (3.40) disappears and (3.22) follows similarly as (3.21). □

Theorem 3.6 also implies an analogue of (3.2) where the total variation distance is replaced by the metric d_{w}. This follows by observing that according to (2.13),

$$\mathsf{E}\Big[d_{\mathrm{w}}(\tilde{\pi}_i^\Delta, \pi_i)\Big] \le \frac{1}{2}\mathsf{E}\left[\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\Big[\big|\tilde{\pi}_i^\Delta \varphi - \pi_i \varphi\big| \,\Big|\, \mathscr{Y}_i\Big]\right].$$

In practice, however, one is typically not interested in evaluating the metric d_{w} which is mostly of theoretical interest. Instead, it is common to take the mean of π_i as the estimate of X_i because it minimises the expectation of the squared Euclidean distance to the true value of the signal [see e.g., 41]. The following corollary establishes the uniform convergence of the approximate posterior mean to the exact posterior mean in the sense of the expected Euclidean distance. For this purpose, we let $\hat{X}_i$ and $\hat{X}_i^\Delta$ denote the means of π_i and $\tilde{\pi}_i^\Delta$, respectively.

Corollary 3.7. *If (A4') and (i), (ii), or (iii) of Theorem 3.6 holds, then there exist c_{12}, $c_{13} > 0$ such that*

$$\mathsf{E}\left[\left\|\hat{X}_i - \hat{X}_i^\Delta\right\|\right] \le c_{12}\exp\left(-c_{13}\Delta^{B_1}\right).$$

Proof. Let $I : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ denote the identity mapping and $I_j : \mathbb{R}^{d_s} \to \mathbb{R}$ the projection to the jth axis. Moreover, we use the shorthand notations $\tilde{X} \triangleq \tilde{h}_i^{-1}(Y_i)$, and $\tilde{X}_j \triangleq I_j(\tilde{X})$. In this case,

$$\mathsf{E}\left[\left\|\hat{X}_i - \hat{X}_i^\Delta\right\|\right] \le \sum_{j=1}^{d_s} \mathsf{E}\left[\left|\pi_i(I_j) - \tilde{\pi}_i^\Delta(I_j)\right|\right].$$

By the triangle inequality and the fact that $\tilde{X}$ is $\mathscr{Y}_i$-measurable, one has

$$\begin{aligned}\left|\pi_i(I_j) - \tilde{\pi}_i^\Delta(I_j)\right| &\le \left|\pi_i(1_{C_i(\Delta)}(I_j - \tilde{X}_j)) - \tilde{\pi}_i^\Delta(1_{C_i(\Delta)}(I_j - \tilde{X}_j))\right| \\ &\quad + \left|\pi_i(1_{\complement C_i(\Delta)}(I_j - \tilde{X}_j))\right| + \left|\tilde{\pi}_i^\Delta(1_{\complement C_i(\Delta)}(I_j - \tilde{X}_j))\right|, \qquad (3.41)\end{aligned}$$

where $\tilde{\pi}_i^\Delta(1_{\complement C_i(\Delta)}(I_j - \tilde{X}_j)) = 0$. According to (A2), for all $x \in C_i(\Delta)$

$$\beta\Delta \ge \beta\left\|Y_i - \tilde{h}_i(x)\right\| \ge \left\|\tilde{h}_i^{-1}(Y_i) - x\right\| \ge \left|\tilde{X}_j - I_j(x)\right|,$$

and thus $1_{C_i(\Delta)}(x)(I_j(x) - \tilde{X}_j) < \beta\Delta$ for all $x \in \mathbb{R}^{d_s}$. Therefore, it follows from (3.41) by taking expectations that

$$\mathsf{E}\left[\left|\pi_i(I_j) - \tilde{\pi}_i^\Delta(I_j)\right|\right] \le \beta\Delta\mathsf{E}\left[\sup_{\|\varphi\|_\infty \le 1} \mathsf{E}\left[\left|\pi_i\varphi - \tilde{\pi}_i^\Delta\varphi\right| \,\middle|\, \mathscr{Y}_i\right]\right] + \mathsf{E}\left[\left|\pi_i(1_{\complement C_i(\Delta)}(I_j - \tilde{X}_j))\right|\right]. \qquad (3.42)$$

For the second term we observe similarly as in (3.15) that

$$\begin{aligned}\mathsf{E}\left[\left|\pi_i(1_{\complement C_i(\Delta)}(I_j - \tilde{X}_j))\right|\right] &\le \mathsf{E}\left[\pi_i\left(1_{\complement C_i(\Delta)}\left\|I - \tilde{X}\right\|\right)\right] \\ &= \mathsf{E}\left[\iint_{\complement D_i(\Delta)} g_{i,y}(x)\left\|x - \tilde{h}_i^{-1}(y)\right\| \mathrm{d}y\, \pi_{i-1}K_i(\mathrm{d}x)\right],\end{aligned}$$

and for the inner integral one has by the change of variables

$$\int_{\complement D_i(\Delta)} g_{i,y}(x) \left\| x - \tilde{h}_i^{-1}(y) \right\| \mathrm{d}y \leq \int_{\|z\|>\Delta-\gamma} \beta(\|z\| + \gamma)\rho_{V_i}(z)\mathrm{d}z$$
$$\leq c\exp\Big((-A_2 + \epsilon)(\Delta - \gamma)^{B_2}\Big) . \tag{3.43}$$

The claim then follows by applying Theorem 3.6 to the first term on the right hand side of (3.42) and (3.43) to the second term. □

3.3 Uniformly convergent sequential Monte Carlo approximation

In this section, we further specify the properties of $\tilde{\pi}^\Delta$ and obtain a general formulation of a feasible filter approximation algorithm which can be parameterised to satisfy (A5) and (A6). The resulting approximation is a sequential Monte Carlo algorithm which consists of an importance sampling step and a resampling scheme. Therefore it can be regarded as a modification of the well known SIR filter introduced in [32]. To avoid confusion, it should be emphasised that in the following, $\tilde{\pi}^\Delta$ is formulated as *an auxiliary particle filter (APF)* type algorithm. The auxiliary particle filter was introduced in [58] and it can be considered as a generalisation of the original SIR filter [see e.g., 36, 35]. The approximation $\tilde{\pi}^\Delta$ is defined as follows:

Definition 3.8. *The filter approximation $\tilde{\pi}^\Delta = (\tilde{\pi}_i^\Delta)_{i\geq 0}$ is a stochastic probability measure valued process satisfying:*

(i) Initialisation: *Define $\tilde{\pi}_0^\Delta \triangleq \pi_0$ and $\{\bar{X}_0^j\}_{j=1}^N$, where $N = N(\Delta) \in \mathbb{N}$, to be a set of independent random variables with the common distribution $\bar{\pi}_0 \in M_\mathrm{P}(\mathbb{R}^{d_s})$ such that*

$$\bar{w}_0(x) \triangleq \frac{\mathrm{d}\pi_0}{\mathrm{d}\bar{\pi}_0}(x) < \bar{W},$$

where $0 < \bar{W} < \infty$. Also, define $\bar{W}_0^j \triangleq \bar{w}_0(\bar{X}_0^j)$.

(ii) Importance sampling: *For all $i > 0$ and $1 \leq j \leq N$, define X_i^j to be a random variable with a distribution $\tilde{K}_i(\bar{X}_{i-1}^j, \cdot)$ such that the Radon-Nikodým derivative*

$$w_i^\Delta(x, y) \triangleq \frac{\mathrm{d}K_i^\Delta(x, \cdot)}{\mathrm{d}\tilde{K}_i(x, \cdot)}(y)$$

exists, is positive for all $x \in C_{i-1}(\Delta)$*, and satisfies* $\sup_{i>0} \left\| w_i^\Delta \right\|_\infty < \infty$*. Also, define* $W_i^j \triangleq w_i^\Delta(\bar{X}_{i-1}^j, X_i^j)$*.*

(iii) Weight update: *For all* $i > 0$*, define*

$$\tilde{\pi}_i^\Delta \triangleq \sum_{j=1}^N \tilde{W}_i^j \delta_{\{X_i^j\}},$$

where $\delta_{\{x\}}$ *denotes the unit point mass located at* $x \in \mathbb{R}^{d_s}$ *and*

$$\tilde{W}_i^j \triangleq \frac{g_i^\Delta(X_i^j) W_i^j \bar{W}_{i-1}^j}{\sum_{\ell=1}^N g_i^\Delta(X_i^\ell) W_i^\ell \bar{W}_{i-1}^\ell}.$$

(iv) Resampling: *For all* $i > 0$*, define* $P_i \triangleq (P_i^1, P_i^2, \ldots, P_i^N)^{\mathrm{T}} \in \mathbb{R}^N$ *such that* $\sum_{j=1}^N P_i^j = 1$ *and*

$$\sup_{\substack{i>0 \\ 0<j\leq N}} \bar{w}_i(X_i^j) < \bar{W},$$

where $\bar{w}_i : \{X_i^j\}_{j=1}^N \to \mathbb{R}_+$ *is defined as* $\bar{w}_i(X_i^j) = \tilde{W}_i^j / P_i^j$*. Moreover, for all* $i > 0$ *and* $1 \leq j \leq N$*, the random variables* $\{\bar{X}_i^j\}_{j=1}^N$ *are defined such that, if* $\zeta_i \triangleq (\zeta_i^1, \zeta_i^2, \ldots, \zeta_i^N)^{\mathrm{T}}$*, where*

$$\zeta_i^j \triangleq \sum_{\ell=1}^N 1_{\{X_i^j\}}(\bar{X}_i^\ell),$$

then there exists $c > 0$ *such that for all* $z = (z_1, z_2, \ldots, z_N)^{\mathrm{T}}$ *where* $\left| z_i \right| \leq 1$*,* $i = 1, \ldots, N$*, one has*

$$z^{\mathrm{T}} \mathsf{E}\left[(\zeta_i - NP_i)(\zeta_i - NP_i)^{\mathrm{T}} \,\middle|\, \mathscr{H}_i \right] z \leq cN. \tag{3.44}$$

Also, define $\bar{W}_i^j \triangleq \bar{w}_i(\bar{X}_i^j)$*.*

The main difference between the conventional APF and $\tilde{\pi}^\Delta$ given in Definition 3.8 is the appearance of the truncation radius Δ in step (ii). Moreover, SMC algorithms are typically parameterised by the sample size, i.e. the number of particles, but here $\tilde{\pi}^\Delta$ is parameterised by the truncation radius Δ, and the sample size N is defined as a function of Δ. The reason for this is that if the function which maps the truncation radius into a sample size is chosen in a certain way, then $\tilde{\pi}^\Delta$ can be shown to satisfy (A5) and (A6) and therefore the approximation is uniformly convergent by Theorem 3.6. It should be noted that a more SIR filter like formulation is obtained by letting $\bar{\pi}_0 = \pi_0$ and by choosing $P_i^j = \tilde{W}_i^j$ for all $i > 0$ and $1 \leq j \leq N$.

In order to implement the approximation $\tilde{\pi}^{\Delta}$ of Definition 3.8 the transition probabilities $\tilde{K}_i$, the resampling probabilities P_i, and the resampling scheme need to be specified. Before doing this, let us show that $\tilde{\pi}^{\Delta}$ indeed can be parameterised to satisfy (A5) and (A6).

Let us define σ-fields $\mathscr{H}_i, \bar{\mathscr{H}}_i \subset \mathscr{F}$ as

$$\mathscr{H}_i \triangleq \sigma\left(Y_m,\, X_n^j,\, \bar{X}_{n-1}^j,\, 1 \le n \le i,\, 1 \le j \le N,\, 0 < m\right)$$

$$\bar{\mathscr{H}}_i \triangleq \sigma\left(Y_m,\, X_n^j,\, \bar{X}_\ell^j,\, 1 \le n \le i,\, 1 \le j \le N,\, 0 \le \ell \le i,\, 0 < m\right).$$

It follows then from Definition 3.8 that $\tilde{\pi}_i^{\Delta}$ is $\mathscr{H}_i$-measurable and according to (iii) $\tilde{\pi}_i^{\Delta}$ is of the form (3.19) where

$$\nu_i^{\Delta} \triangleq \frac{1}{N}\sum_{j=1}^{N} W_i^j \bar{W}_{i-1}^j \delta_{\{X_i^j\}}. \tag{3.45}$$

According to (ii) and (iv), $w_i^{\Delta} > 0$, P-a.s. and thus (A5) is satisfied. Let us then consider (A6). Because $\mathscr{H}_{i-1} \subset \bar{\mathscr{H}}_{i-1}$, we can write for all $i > 0$

$$\begin{aligned}\mathsf{E}\left[\left|\tilde{\pi}_{i-1}^{\Delta} K_i^{\Delta}\varphi - \nu_i^{\Delta}\varphi\right| \,\middle|\, \mathscr{H}_{i-1}\right] \le{}& \mathsf{E}\left[\mathsf{E}\left[\left|\bar{\pi}_{i-1}^{\Delta} K_i^{\Delta}\varphi - \nu_i^{\Delta}\varphi\right| \,\middle|\, \bar{\mathscr{H}}_{i-1}\right] \,\middle|\, \mathscr{H}_{i-1}\right] \\ &+ \mathsf{E}\left[\left|\tilde{\pi}_{i-1}^{\Delta} K_i^{\Delta}\varphi - \bar{\pi}_{i-1}^{\Delta} K_i^{\Delta}\varphi\right| \,\middle|\, \mathscr{H}_{i-1}\right],\end{aligned} \tag{3.46}$$

where

$$\bar{\pi}_i^{\Delta} \triangleq \frac{1}{N}\sum_{j=1}^{N} \bar{W}_i^j \delta_{\{\bar{X}_i^j\}}, \tag{3.47}$$

for all $i \ge 0$. This random probability measure is the approximation of π_i after the resampling step. Because

$$\begin{aligned}(\bar{\pi}_{i-1}^{\Delta} K_i^{\Delta}\varphi - \nu_i^{\Delta}\varphi)^2 ={}& \frac{1}{N^2}\sum_{j=1}^{N}\left(\bar{W}_{i-1}^j K_i^{\Delta}\varphi(\bar{X}_{i-1}^j) - W_i^j \bar{W}_{i-1}^j \varphi(X_i^j)\right)^2 \\ &+ \frac{1}{N^2}\sum_{j\neq\ell} \bar{W}_{i-1}^j \bar{W}_{i-1}^\ell \left(K_i^{\Delta}\varphi(\bar{X}_{i-1}^j) - W_i^j\varphi(X_i^j)\right)\left(K_i^{\Delta}\varphi(\bar{X}_{i-1}^\ell) - W_i^\ell\varphi(X_i^\ell)\right),\end{aligned}$$

and because

$$\mathsf{E}\left[W_i^j\varphi(X_i^j) \,\middle|\, \bar{\mathscr{H}}_{i-1}\right] = \int \varphi(x) w_i^{\Delta}(\bar{X}_{i-1}^j, x)\tilde{K}_i(\bar{X}_{i-1}^j, \mathrm{d}x) = K_i^{\Delta}\varphi(\bar{X}_{i-1}^j),$$

it follows from the conditional independence of $\{X_i^j\}_{j=1}^N$ given $\hat{\mathscr{H}}_{i-1}$ that

$$
\begin{aligned}
&\mathsf{E}\left[\,(\bar{\pi}_{i-1}^{\Delta}K_i^{\Delta}\varphi - \nu_i^{\Delta}\varphi)^2 \,\middle|\, \bar{\mathscr{H}}_{i-1}\right] \\
&= \frac{1}{N^2}\sum_{j=1}^{N}\left(\bar{W}_{i-1}^j\right)^2\left(\mathsf{E}\left[\left(W_i^j\varphi(X_i^j)\right)^2 \,\middle|\, \bar{\mathscr{H}}_{i-1}\right] - \left(K_i^{\Delta}\varphi(\bar{X}_{i-1}^j)\right)^2\right) \\
&\leq \frac{1}{N^2}\sum_{j=1}^{N}\left(\bar{W}_{i-1}^j\right)^2\int\left(\varphi(x)w_i^{\Delta}(\bar{X}_{i-1}^j,x)\right)^2\tilde{K}_i(\bar{X}_{i-1}^j,\mathrm{d}x) \leq \bar{W}^2\frac{\left\|w_i^{\Delta}\varphi\right\|_{\infty}^2}{N},
\end{aligned}
$$

and finally by Jensen's inequality

$$
\mathsf{E}\left[\left|\bar{\pi}_{i-1}^{\Delta}K_i^{\Delta}\varphi - \nu_i^{\Delta}\varphi\right| \,\middle|\, \bar{\mathscr{H}}_{i-1}\right] \leq \frac{\bar{W}\left\|w_i^{\Delta}\varphi\right\|_{\infty}}{\sqrt{N}}. \tag{3.48}
$$

For the second term in the right hand side of (3.46), one can check that if $i = 0$, then

$$
\mathsf{E}\left[\left|\tilde{\pi}_i^{\Delta}\varphi - \bar{\pi}_i^{\Delta}\varphi\right| \,\middle|\, \mathscr{H}_i\right] \leq \frac{\bar{W}}{\sqrt{N}}.
$$

For all $i > 0$, we define $\tilde{W}_i \triangleq (\tilde{W}_i^1, \tilde{W}_i^2, \ldots, \tilde{W}_i^N)^{\mathrm{T}}$, $\Phi_i \triangleq (\varphi(X_i^1), \varphi(X_i^2), \ldots, \varphi(X_i^N))^{\mathrm{T}}$ and $\bar{W}_i \in \mathbb{R}^{N\times N}$ as a diagonal matrix with the elements $\bar{w}_i(X_i^1), \bar{w}_i(X_i^2), \ldots, \bar{w}_i(X_i^N)$ on the diagonal. In this case, $\tilde{\pi}_i^{\Delta}\varphi = \tilde{W}_i^{\mathrm{T}}\Phi_i$, $\bar{\pi}_i^{\Delta}\varphi = \frac{1}{N}\zeta_i^{\mathrm{T}}\bar{W}_i^{\mathrm{T}}\Phi_i$, $\tilde{W}_i = \bar{W}_i P_i$, and thus according to (iv), there exists $c \geq 1$ such that

$$
\begin{aligned}
\mathsf{E}\left[\left(\tilde{\pi}_i^{\Delta}\varphi - \bar{\pi}_i^{\Delta}\varphi\right)^2 \,\middle|\, \mathscr{H}_i\right] &= \frac{1}{N^2}\Phi_i^{\mathrm{T}}\mathsf{E}\left[(\bar{W}_i\zeta_i - N\tilde{W}_i)(\bar{W}_i\zeta_i - N\tilde{W}_i)^{\mathrm{T}} \,\middle|\, \mathscr{H}_i\right]\Phi_i \\
&= \frac{1}{N^2}\Phi_i^{\mathrm{T}}\bar{W}_i\mathsf{E}\left[(\zeta_i - NP_i)(\zeta_i - NP_i)^{\mathrm{T}} \,\middle|\, \mathscr{H}_i\right]\bar{W}_i^{\mathrm{T}}\Phi_i \leq \frac{\bar{W}^2 c}{N},
\end{aligned}
$$

where the inequality follows from the fact that $\varphi(X_i^j)\bar{W}_i^j/\bar{W} \leq 1$. Again by Jensen's inequality it follows that for all $i \geq 0$

$$
\mathsf{E}\left[\left|\tilde{\pi}_i^{\Delta}\varphi - \bar{\pi}_i^{\Delta}\varphi\right| \,\middle|\, \mathscr{H}_i\right] \leq \frac{\sqrt{c}\bar{W}}{\sqrt{N}}. \tag{3.49}
$$

Because $\sup_{i>0}\sup_{\|\varphi\|_{\infty}\leq 1}\left\|K_i^{\Delta}\varphi\right\|_{\infty} \leq 1$ and according to (ii) $\sup_{i>0}\left\|w_i^{\Delta}\right\|_{\infty} < \infty$, it follows by the substitution of (3.48) and (3.49) into (3.46) that there exists $c' = c'(\Delta) > 0$ such that

$$
\sup_{\|\varphi\|_{\infty}\leq 1}\mathsf{E}\left[\left|\tilde{\pi}_{i-1}^{\Delta}K_i^{\Delta}\varphi - \nu_i^{\Delta}\varphi\right| \,\middle|\, \mathscr{H}_{i-1}\right] \leq \frac{c'(\Delta)}{\sqrt{N}}, \tag{3.50}
$$

for all $i > 0$. Consequently, (A6) holds if

$$N(\Delta) \geq \left(\frac{c'(\Delta)}{M_4}\right)^2 \exp\left(2a_4\Delta^{b_2}\right). \tag{3.51}$$

In conclusion, the particle filter of Definition 3.8 is uniformly convergent if the sample size N increases sufficiently fast as $\Delta \to \infty$.

3.3.1 Resampling scheme

The conditions imposed on the resampling scheme in (iv) of Definition 3.8 are similar to those given in [15] with the exception that here the resampling algorithm is applied to the probabilities P_i^j that in general are allowed to be different from the weights $\tilde{W}_i^j$. This difference enables the APF type formulation. Following the typical terminology in the particle filtering related literature, the interpretation of the random variable ζ_i^j is that it equals the number of offspring produced by the jth particle at time i. If one chooses $P_i = \tilde{W}_i$, then (iv) implies that the numbers ζ_i^j are approximately proportional to the weights $\tilde{W}_i$, i.e.

$$\tilde{\pi}_i^\Delta \approx \frac{1}{N}\sum_{j=1}^N \zeta_i^j \delta_{\{X_i^j\}}.$$

If $P_i \neq \tilde{W}_i$, then the numbers of particle duplicates are approximately proportional to P_i, i.e.

$$P \triangleq \sum_{j=1}^N P_i^j \delta_{\{X_i^j\}} \approx \frac{1}{N}\sum_{j=1}^N \zeta_i^j \delta_{\{X_i^j\}}.$$

According to the principle of the importance sampling, ζ_i can still be used for approximating $\tilde{\pi}_i^\Delta$ if the numbers ζ_i^j are compensated by appropriate weights. These weights should be equal to the Radon-Nikodým derivative $\mathrm{d}\tilde{\pi}_i^\Delta/\mathrm{d}P$ which is precisely the function $\bar{w}_i$ defined in (iv) and therefore the approximation $\bar{\pi}_i^\Delta$ given in (3.47) is obtained.

The simplest resampling scheme satisfying (iv) is *the multinomial resampling* method. In this case ζ_i is a random variable with a multinomial distribution, i.e.

$$\mathsf{P}(\zeta_i^1 = n_1, \zeta_i^2 = n_2, \ldots, \zeta_i^N = n_N) = \frac{N!}{n_1!n_2!\cdots n_N!}(\tilde{W}_i^1)^{n_1}(\tilde{W}_i^2)^{n_2}\cdots(\tilde{W}_i^N)^{n_N}.$$

The proof that the multinomial resampling scheme satisfies (3.44) is simple and can be found, e.g. in [15, page 28]. In practice, the multinomial resampling algorithm is implemented by letting $\bar{X}_i^j$, $1 \leq j \leq N$ be an independent random variable with the distribution $\tilde{\pi}_i^\Delta$.

Another resampling scheme which satisfies (iv) is *the tree based branching algorithm (TBBA)* described in [15]. The proof that TBBA satisfies (3.44) is somewhat more involved but can also be found in [15].

Several other resampling schemes have been proposed in the literature as well. *The stochastic universal sampling* described already in [6] has been later proposed to be used as a resampling method in SMC algorithms [see e.g., 13, 3]. Very similar, but entirely deterministic resampling scheme was described earlier in the SMC context in [42] under the name *deterministic resampling*. Also the so called *stratified resampling* scheme was proposed in [42]. Regarding the stratification, it should be noted that the stratified resampling algorithm described in [42] represents only one possible stratification but in general stratification can be done in a number of ways [see e.g., 14]. For more details on the choice of the stratification, see e.g. [35]. Another commonly used resampling scheme is *the residual resampling* method described in [48]. In theory, $\tilde{\pi}^\Delta$ is uniformly convergent for any resampling method which can be shown to satisfy the conditions imposed in (iv) of Definition 3.8. Of all the resampling methods mentioned above this can be done at least for the multinomial resampling method and the TBBA.

3.3.2 Importance distribution

The only essential difference between $\tilde{\pi}^\Delta$ and the conventional APF is the requirement imposed by the assumption (A5) that the importance distribution specified by the transition probabilities $\tilde{K}_i$ must assign zero probability to the set $\complement C_i(\Delta)$. To see that this is not the case with the conventional APF in general, it suffices to consider a signal model with Gaussian noise. As a general rule for specifying an importance distribution satisfying (A5), we define for all $x \in \mathbb{R}^{d_s}$

$$\tilde{K}_i(x, \cdot) = \mathsf{P}_{\tilde{h}_i^{-1}(Z)}(\cdot), \tag{3.52}$$

where $\mathsf{P}_{\tilde{h}_i^{-1}(Z)}$ denotes the distribution of the random variable $\tilde{h}_i^{-1}(Z)$, and Z is a random variable with a uniform distribution on the Y_i centered ball of radius Δ. This choice of $\tilde{K}_i$ obviously satisfies (A5). Moreover, by assuming that for all $x \in \mathbb{R}^{d_s}$, one has $|\det(\tilde{h}_i'(x))| > 0$, where $\tilde{h}_i'(x)$ denotes the Jacobian matrix of $\tilde{h}_i$ at $x \in \mathbb{R}^{d_s}$, the density $\rho_{X_i^j}$ of the random variable X_i^j with respect to λ_{d_s} satisfies

$$\rho_{X_i^j}(x) = 1_{C_i(\Delta)}(x)|\det \tilde{h}_i'(x)|/\tilde{V}_{d_s}(\Delta).$$

According to (A2),

$$\left|\det \tilde{h}_i'(x)\right|^{-1} = \det(\tilde{h}_i'(x))^{-1} = \det(\tilde{h}_i^{-1})'(\tilde{h}_i(x)) \leq d_s!\beta^{d_s},$$

and thus,

$$\sup_{i>0} \left\| w_i^\Delta \right\|_\infty = \sup_{\substack{x,y \in \mathbb{R}^{d_s} \\ i>0}} \frac{\tilde{V}_{d_s}(\Delta) 1_{C_i(\Delta)}(y) k_i(x,y)}{|\det \tilde{h}_i'(y)|} \leq c'' \Delta^{d_s}, \tag{3.53}$$

for some $c'' > 0$. The substitution of this approximation into (3.51) yields a lower bound for the sample size $N(\Delta)$ which ensures (A6).

The importance distribution described above is specified by the transition probability $\tilde{K}_i$ which is independent of its first argument. This is fairly uncommon in APF algorithms as it is more common to have, e.g. $\tilde{K}_i = K_i$ which however does not satisfy (A5). On the other hand, it has been acknowledged in the SMC related literature that the importance distribution should be *adapted*. This terminology is taken from [58], and it means that the importance distribution should depend on the latest observation. This is precisely what happens with the transition probabilities $\tilde{K}_i$ described above, as the random variables X_i^j are generated in the compact neighborhood of the preimage of Y_i. To be able to do this, we pay the price of assuming the existence of the bijective mapping $\tilde{h}_i$ which in many practical applications does not exist. It should also be noted that the given importance distribution only requires the ability to evaluate $\tilde{h}_i^{-1}$ and simulate a random variable on a d_s-dimensional unit ball. This random sample generation is feasible in high dimensions without resorting to *the rejection method*. Details on generating these random variables can be found e.g. in [24, Theorem 4.3, page 229].

3.4 Numerical experiments

In this section, Theorem 3.6 and Corollary 3.7 are illustrated by some numerical experiments. Two simple filter frameworks are considered and the particle filter of Definition 3.8 employing the multinomial resampling scheme and the importance distribution proposed in Section 3.3.2 is applied to both of the filter frameworks.

In the first experiment a linear-Gaussian model is considered. It is well known that in this case for all $i > 0$, π_i is equal to a normal distribution whose mean and covariance can be computed exactly using the Kalman filter recursion. Therefore, numerical approximations of π in this case are not of any practical interest. On the other hand, the possibility to compute π exactly enables us to compute also the error of the approximation $\tilde{\pi}^\Delta$ exactly. This is the motivation for considering the linear-Gaussian case.

In the second experiment, the uniformly convergent approximation is applied to a nonlinear model with Gaussian noise. In this case, the filter and therefore the approximation error cannot be computed exactly and therefore a conventional SIR filter with a large sample size is used as a reference to which the solution of the uniformly convergent approximation is compared.

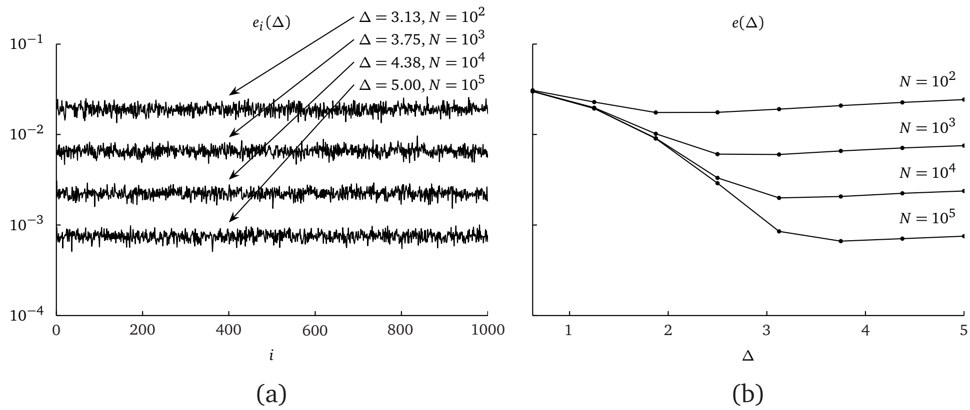

Figure 3.2: Approximate mean errors and average mean errors for the linear-Gaussian model.

Although both of the above mentioned frameworks assume Gaussian noise distributions it should be emphasised that this is not required by Theorem 3.6. Indeed, the distributions could be also e.g. mixtures of Gaussian distributions, double exponential distributions or convolutions of these distributions with distributions that have a bounded support.

3.4.1 Linear model

Let $\mathrm{N}(x, y)$ denote a normal distribution with mean x and covariance y. Suppose that $X_0 \sim \mathrm{N}(0, 1)$ and for all $i > 0$

$$X_i = X_{i-1} + W_i$$
$$Y_i = 4X_i + V_i,$$

where $V_i \sim \mathrm{N}(0, 1)$ and $W_i \sim \mathrm{N}(0, 2)$ independently for all $i > 0$. In this case, X is a nonergodic and time homogenous Markov chain [see e.g., 50, pages 311-316] and it can be shown that this model satisfies the conditions of Theorem 3.6.

The approximation error is considered in the sense of Corollary 3.7. Although the Euclidean distance between the approximate and the exact mean can be computed exactly in the linear-Gaussian case, the computation of the expected value of the distance is intractable. Therefore the expected error was approximated using the Monte Carlo method, i.e.

$$\mathsf{E}\left[\left\|\hat{X}_i - \hat{X}_i^{\Delta}\right\|\right] \approx e_i(\Delta) \triangleq \frac{1}{N_{\text{obs}}} \sum_{j=1}^{N_{\text{obs}}} \left\|\hat{X}_i^j - \hat{X}_i^{\Delta,j}\right\|,$$

where $\hat{X}_i$ and $\hat{X}_i^{\Delta}$ denote the exact and the approximate posterior mean at time i, respectively, and $\hat{X}_i^j$ and $\hat{X}_i^{\Delta,j}$ denote the jth realisation of the exact and the approximate posterior

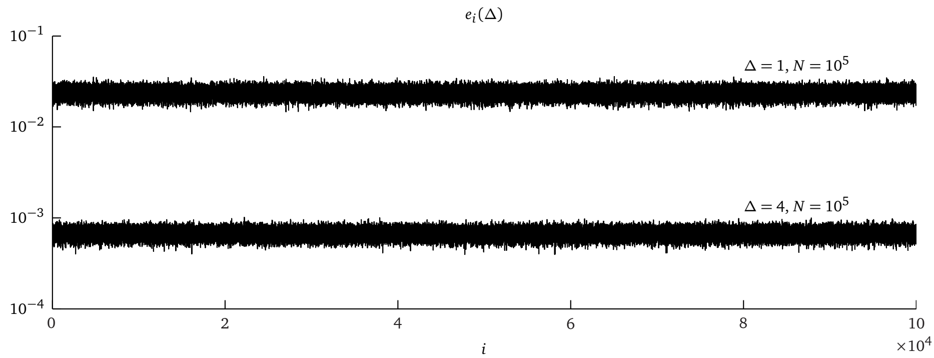

Figure 3.3: Approximate mean errors for the linear-Gaussian model.

mean at time i, respectively. Moreover, $N_{\text{obs}} \in \mathbb{N}$ denotes the number of data sets used for approximating the expected distance.

Instead of specifying the sample size N as a specific function of Δ, the numerical experiments were done for all $(\Delta, N) \in \mathscr{D} \times \mathscr{N}$ where

$$\mathscr{N} = \{10^2, 10^3, 10^4, 10^5\}$$
$$\mathscr{D} = \{0.625,\ 1.250,\ 1.875,\ 2.500,\ 3.125,\ 3.750,\ 4.375,\ 5.000\}.$$

The set $\mathscr{N}$ was chosen such that the computational cost of evaluating $\tilde{\pi}_i^\Delta$ was reasonable and the definition of $\mathscr{D}$ is based on experiments and it was chosen such that the effect of Δ and N is well illustrated by the experiments.

Figure 3.2 shows the results of the experiment. In Figure 3.2(a) the approximate mean errors for $N_{\text{obs}} = 50$ are illustrated on the time interval $1 \le i \le 1000$ for four different pairs (Δ, N). The results are consistent with Theorem 3.6 and Corollary 3.7 as $e_i(\Delta)$ appears to be nearly independent of i and therefore uniformly bounded in time. Moreover, this uniform bound decreases as Δ and N are increased. Figure 3.2(a) suggests that the time average of $e_i(\Delta)$, i.e.

$$e(\Delta) \triangleq \frac{1}{T} \sum_{i=1}^{T} e_i(\Delta),$$

where T is the length of the simulation, is a relatively good approximation of the uniform bound for the expected error. Figure 3.2(b) shows the approximate average mean errors for all $(\Delta, N) \in \mathscr{D} \times \mathscr{N}$.

It is observed that the twofold construction of the approximation $\tilde{\pi}^\Delta$ can be seen in Figure 3.2(b) as for $\Delta < 3$, the average error $e(\Delta)$ appears to reach a level which cannot be improved by increasing the sample size. This level represents the error of the truncated filter

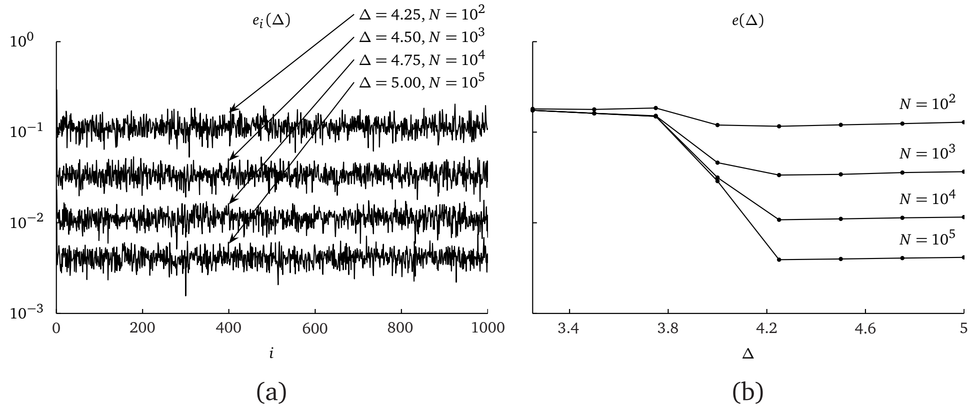

Figure 3.4: Approximate mean errors and average mean errors for the nonlinear model.

π^Δ and therefore the approximation $\tilde{\pi}^\Delta$, which after all is an approximation of π^Δ rather than π, cannot outperform π^Δ. Also, it is observed that in order to ensure the convergence to zero, it is not sufficient to increase only Δ and keep N fixed. This is illustrated in Figure 3.2(b) as for each fixed value of N the average error curve appears to be increasing for sufficiently large values of Δ.

Because the study of uniform convergence is motivated by the interest in the behaviour of the error for long time intervals, the filter in the linear-Gaussian case was also approximated on the time interval $0 < i \leq 10^5$. The results of this experiment are illustrated in Figure 3.3 and they appear to be in accordance with Theorem 3.6 as well.

3.4.2 Nonlinear model

Suppose that $X_0 \sim \mathrm{N}(0, 1)$ and for all $i > 0$

$$X_i = \frac{1}{2}X_{i-1} + \frac{25X_{i-1}}{1+X_{i-1}^2} + 8\cos(1.2i) + W_i$$
$$Y_i = 4X_i + 4\sin(2X_i) + V_i,$$

where $V_i \sim \mathrm{N}(0, 0.006)$ and $W_i \sim \mathrm{N}(0, 2)$, independently. This signal model is adapted from the popular example of [42]. In this case, π cannot be computed exactly because of the nonlinearity of f_i and h_i. Therefore, in order to approximate the error, a conventional SIR filter with sample size $N = 10^6$ was used as a reference. The implemented SIR filter used the signal transition kernel for the importance distribution and it employed the multinomial resampling scheme. The experiment was otherwise similar to the linear case, except that this time we defined

$$\mathscr{D} = \{3.25, 3.50, 3.75, 4.00, 4.25, 4.50, 4.75, 5.00\}.$$

Again, this choice of $\mathscr{D}$ was based on the experiments and it was chosen to provide a good illustration of the effect of Δ and N. The results of the experiment are shown in Figure 3.4.

Figure 3.4(a) shows the approximate mean errors for four different pairs (Δ, N) and Figure 3.4(b) shows the approximate average mean errors for all $(\Delta, N) \in \mathscr{D} \times \mathscr{N}$. Also for the nonlinear case, the results appear to be in accordance with Theorem 3.6. In Figure 3.4(b), the average error decreases rapidly for all values of N when $\Delta \approx 4$. This phenomenon can be explained by the bounded component $\bar{h}_i(x) = 4\sin(2x)$ in the observation model. Roughly speaking, the function $\bar{h}_i$ causes the likelihood function g_i to be multimodal but the distance between the modes is bounded. When $\Delta \approx 4$ or greater, all the modes are included in the set $C_i(\Delta)$ but for $\Delta < 4$ some of the modes may remain outside $C_i(\Delta)$ and therefore the error is large. Moreover, it should be noted that according to Figure 3.4(b), the error for $\Delta < 4$ appears to be due to the truncation as the average error is nearly independent of the sample size N.

According to Definition 3.8 the computational cost of the uniformly convergent particle filter should be approximately the same as for the conventional SIR filter. Some extra cost may of course be introduced by the requirement that the samples are simulated inside the set $C_i(\Delta)$ but for example in the nonlinear framework described above, the evaluation of the uniformly convergent particle filter was approximately 1.2 times the time of the SIR filter with the same sample size.

Chapter 4

Conclusions

This thesis has addressed two important problems related to the stochastic discrete time filters, namely, the stability of the filter with respect to its initial conditions and the uniform convergence of certain filter approximations. In this chapter, the main results of this work are reviewed accompanied by some discussion about the conclusions of the results.

This chapter is organised as follows. Section 4.1 consists of discussion about the general conclusions regarding the stability results and in Section 4.2 a similar discussion about the uniform convergence results is given. Finally, some topics for future research are pointed out in Section 4.3.

4.1 Stability

Regarding the stability of the discrete time filter, it was shown that with relatively weak assumptions on the signal process the filter is stable provided that the observation geometry is good enough and that the tails of the observation noise distributions are sufficiently light compared to the tails of the signal noise distributions. The sufficiently good observation geometry in this case means the conditions of the assumption (A2). Roughly speaking, the existence of the bijective component $\tilde{h}_i$ implies that the observations carry information about all dimensions of the state space. Moreover, the assumption that $\tilde{h}_i^{-1}$ is uniformly Lipschitz can be interpreted to mean that the information content of the observations is bounded from below. If the Lipschitz coefficient β was allowed to increase, the observations might eventually tell nothing about the state of the signal. Consider for example a constant function $\tilde{h}_i$ independent of X_i. To some extent the assumption (A2) can be regarded also as an analog of the observability condition in the stability analysis of the linear filters. Unfortunately, it should be acknowledged that many interesting practical applications do not satisfy the assumption (A2).

Perhaps the most important conclusion regarding the stability is that for the observation model

$$Y_i = h_i(X_i) + aV_i, \qquad a \in (0, \infty)$$

it is not in general necessary that a be small. Indeed, Theorem 2.15 states that the filter is stable for arbitrarily large a provided that the tails of the observation noise are sufficiently light compared to the tails of the signal noise distributions. Only in the case that the tails are equally heavy is a required to be small. This observation was originally made in [56] but the result has been further extended in this work. The approach used here also provided explicit rates for the convergence of the error in the almost sure sense. Unfortunately, these convergence rates are not exponential and therefore a comparison with the results given in [12] suggests that the rates given by Corollary 2.16 are not optimal.

The majority of the literature regarding the filter stability is involved with proving the sufficiency of certain conditions for the stability but it is equally important to establish necessary conditions as well. To the author's knowledge, the only general result on the necessity of conditions is the stabilisability and detectability conditions for the mean and the covariance process of the linear filter. However, the stability of the mean and the covariance process is not equivalent to the stability of the corresponding probability measure valued process in the total variation distance. For example, two random walks starting from different initial values have a time invariant mean which is equal to the initial value. Therefore the mean process is not stable but because of the increasing covariance the total variation distance between the two processes vanishes. Therefore the stabilisability and detectability are not necessary for the stability in total variation.

It is a simple task to give an example of an unstable filter in some degenerate case, but for more interesting applications the task is challenging. It is natural to ask if the analysis given in Chapter 2 could be extended to obtain some necessary conditions as well. Unfortunately, this seems to be impossible. The reason for this is that most of the analysis consists of deriving upper bounds for the error, but in order to find necessary conditions one is in fact interested in finding lower bounds. Therefore the majority of the analysis is not of any interest when proving necessary conditions. Already the starting point of the analysis, i.e. the fundamental idea of using the Dobrushin ergodic coefficient is problematic because only an upper bound for the distance between the images of the Markov operation is given by the ergodic coefficient and for the necessity one would need a lower bound. Therefore it seems that in order to prove necessary conditions for the stability, entirely original ideas are needed.

4.2 Uniform convergence

Chapter 3 focused on proving the uniform convergence of filter approximations. The approach was very similar to the one in [56] but the analysis was extended to provide general sufficient conditions for the uniform convergence of filter approximations. These conditions were given by (A5) and (A6). Roughly speaking, (A5) implies that the approximating algorithm has a certain structure and (A6) implies that the approximation has some convergence properties. To be more specific, it follows from (A6) that for each time step the prediction distribution of the approximate filter, i.e. $\tilde{\pi}_{i-1}^{\Delta} K_i^{\Delta}$ is approximated well enough. To some extent, (A6) is similar to the conditions given in [15].

Moreover, it was shown that the conventional SIR filter or the APF with some simple modifications satisfy (A5) and (A6) if the sample size N is defined as a sufficiently fast increasing function of the truncation radius Δ. More explicitly, it was shown that there exist $c, c' > 0$ such that if

$$N(\Delta) \geq c\exp\left(c'\Delta^{b_2}\right),$$

then $\tilde{\pi}^{\Delta}$ is uniformly convergent. Because the computational cost of the SIR filter is determined by the sample size, this lower bound for the sample size can be substituted into the convergence rate provided by Theorem 3.6 yielding a convergence rate of the form $c\exp\left(-c'(\ln N/c'')^{B_1/b_2}\right)$ for some $c, c', c'' > 0$. This rate is the effective rate of convergence in the sense that it represents the error as a function of N which in turn represents the computational cost of evaluating the approximation. In particular, it is observed that in the case $B_1 = b_2$ the convergence rate is of the form $cN^{-c'}$ for some $c, c' > 0$. It is natural to ask for a faster rate of convergence but it should also be kept in mind that this convergence rate is derived for an algorithm based on the Monte Carlo method and therefore it is not expected to outperform the convergence rate of the classical Monte Carlo integral.

All the constants required for computing numerical bounds for the approximation error of the uniformly convergent particle filter can be evaluated or at least approximated. However, the resulting bound for the error is expected to be unnecessarily loose and therefore Theorem 3.6 and Corollary 3.7 cannot be used for obtaining reasonable bounds for the errors in practice.

In order to implement the uniformly convergent particle filter described in Section 3.3 the importance distributions need to be defined in such a manner that the conditions of Definition 3.8 are satisfied. An example of such a choice of importance distributions was given in Section 3.3.2 but also the rejection method can be used for drawing samples from the set $C_i(\Delta)$. In this case, the computational cost of the algorithm becomes random and therefore the uniform convergence can be obtained only with respect to the expected computational cost, provided that the rejection rate is uniformly bounded with respect to time.

A uniform bound for the rejection rate can be ensured e.g. if the rejection method is used for drawing samples from the Y_i centered ball with radius Δ instead of $C_i(\Delta)$ directly. This is because the randomness of the algorithm in this case is entirely due to the rejection method and independent of the observations. If the samples are generated according to the signal transition probabilities and the rejection method is used for obtaining a sample from $C_i(\Delta)$, then the rejection rate and thus the expected computational cost also depend on the observations. In this case, the uniform bound for the rejection rate has to be proved separately. This proof has not been given in this work or in [56].

4.3 Research directions

So far the stability of the filter has been established in the literature under three different types of conditions: sufficiently well behaved signal, sufficiently accurate observations, or sufficiently light tailed observation noise. None of these conditions are necessary. It is a challenging problem of great interest to obtain a general result stating what sort of conditions are necessary for the stability of the filter.

The uniform convergence was considered in this work in the mean sense. From the practical point of view it would be of greater interest to establish the convergence in the almost sure sense. Although the problem has not been addressed in this work, it seems reasonable to believe that uniform convergence in the almost sure sense cannot be obtained for SIR filters. However, it also seems plausible that uniform convergence can be obtained for a sample size with a modest growth. Modest in this case means for example logarithmic. This statement is of course speculation and its verification, if possible, is left for future research.

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