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Abstract

The present work extends known finite-dimensional constrained optimal control realizations to the realm of well-posed regular linear infinite-dimensional systems modelled by partial differential equations. The structure-preserving Cayley-Tustin transformation is utilized to approximate the continuous-time system by a discrete-time model representation without using any spatial discretization or model reduction. The discrete-time model is utilized in the design of model predictive controller accounting for optimality, stabilization, and input and output/state constraints in an explicit way. The proposed model predictive controller is dual-mode in the sense that predictive controller steers the state to a set where exponentially stabilizing unconstrained feedback can be utilized without violating the constraints. The construction of the model predictive controller leads to a finite-dimensional constrained quadratic optimization problem easily solvable by standard numerical methods. Two representative examples of partial differential equations are considered.

Key words: infinite-dimensional systems, modeling and control optimization, controller constraints and structure, model predictive control, regular linear systems, Cayley-Tustin transform

1 Introduction

The concept of *regular linear systems* came about at the turn of 1990's by the work of George Weiss [33–35]. This subclass of abstract linear systems is essentially the Hilbert space counterpart of the finite-dimensional systems described by the state-space equations:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

where, however, the operators A, B and C may be unbounded. Regular linear systems are often encountered in the study of partial differential equations (PDEs) with boundary controls and boundary observations, and they cover a large class of abstract systems of practical interest.

The control of linear distributed parameter systems (DPS) is a mature control field with seminal contributions given in [5, 13, 26, 30, 31]. The system theoretic properties and controller designs were explored in these contributions with the emphasis on full state feedback, boundary and/or in-domain stabilization, optimality and robustness. In addition, classical control problems such as state feedback regulation [21] and robust output regulation [22, 23] have been considered, and regulator theory has been developed for regular linear systems. Above contributions fully explored the functional space setting of the continuous-time system representation and only minor considerations have been devoted to the discrete-time counterparts. In addition, despite the myriad of work on unconstrained stabilization, the design of low order constrained optimal/suboptimal controllers for DPS which accounts for input and state/output constraints remained elusive.

Over the past decade, there have been several attempts to address control of distributed parameter systems within an input and/or state constrained optimal control setting. There are several works on dynamical analysis and optimal control of hyperbolic PDEs, most notably the work of Aksikas et al. on optimal linear quadratic feedback controller design for hyperbolic DPS. [1, 2, 29]. Other contributions considered optimal and model predictive control applied to Riesz spectral systems (parabolic and higher order dissipative PDEs) with a separable eigenspectrum of the underlying dissipative spectral operator and successfully designed algorithms that account for the input and state con-

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straints [6, 14, 40]. In prior contributions, some type of spatial approximation is applied to the PDE models to arrive at finite-dimensional models utilized in the controller design. As it will be claimed and demonstrated in the subsequent sections, the linear distributed parameter system can be treated intact and controller design can be accomplished without any spatial model approximation or reduction.

The research area of model predictive control (MPC) and contributions associated with this design methodology has flourished over past two decades [8, 15, 16, 24]. The appealing nature of applying to the state the first control input in a finite sequence of control inputs obtained as a solution of an online constrained, discrete-time, optimal control problem with explicit account for the control and state constraints, and achieving stability by adding a terminal cost or terminal constraints, or by extending the horizon of the the optimal control problem, is well understood and explored [15,16,25] but could not be easily extended to the DPS setting. Apart from the aforementioned contributions [6, 14, 40] where some type of model approximation has been applied, other contributions explored unconstrained MPC with emphasis on the computational complexity of the optimization problem [7]. However, the clear link between the discrete constrained optimization based MPC design, the well-understood modelling of distributed parameter systems described by PDEs, and the well-established control theory of linear DPS has not yet been established apart from the recent work by the authors [11, 39].

Motivated by the preceding, in this contribution, the model predictive control for regular linear systems is developed. In particular, the essential feature of the discrete-time infinite-dimensional representation necessary in the MPC design preserving the continuous-time system properties is established by applying the Caley-Tustin (CT) [10] time discretization, implying that no spatial discretization or model reduction is required. At the core of the CT transformation, one can find the application of a Crank-Nicolson type time discretization scheme which is a well-know implicit midpoint integration rule that is symmetric, symplectic (Hamiltonian preserving) [9], and guarantees structure preserving numerical integration so that stability and controllability are not altered by the discrete-time infinite-dimensional model representation. Furthermore, boundary and/or point actuation transformed to the discrete-setting yields bounded operators.

As the first main contribution, the MPC design utilized in [39] is generalized for stable regular linear systems (Theorem 2). Under the assumption of infinite-time admissibility of the observation operator, optimality and stability of the proposed design is proved. The design is demonstrated on a numerical example of the one-dimensional wave equation.

As the second main contribution, an MPC-based control design is presented to achieve constrained stabilization of exponentially stabilizable systems (Theorem 4) and the design is demonstrated on a simulation study of a tubular reactor. The proposed design belongs to the class of dual-mode control [17, 28] implying that the model predictive controller steers the state to the neighborhood of the origin where local unconstrained stabilizing feedback can be applied without violating the input constraints. A stabilizing terminal penalty is added to the MPC formulation to guarantee stabilizability while no terminal constraints are imposed. Stabilization of a finite-number of unstable eigenvalues is considered in the MPC setting in [39], but here the proposed methodology can be applied to arbitrary exponentially stabilizable systems. Finally, the proposed work provides a foundation to link regular linear systems to the well-established area of linear model predictive control designs.

The structure of the paper is as follows. In Section 2, we present the notation, the mathematical preliminaries concerning regular linear systems and the Cayley-Tustin time discretization scheme. In Section 3, we present the MPC problem, and in Sections 3.1 and 3.2, stability and optimality results of the proposed MPC and dual-mode control designs are presented. In Section 4, we present, as an example of a stable system, the wave equation on a one-dimensional spatial domain and compute the operators corresponding to the Cayley-Tustin transform and their adjoints. Furthermore, in Section 4.3, we derive a solution of the Lyapunov equation for the wave equation as required by the proposed MPC design. The performance of the MPC is demonstrated by numerical simulations of the controlled wave equation in Section 4.4. In Section 5, the dual-mode controller design is demonstrated on an unstable tubular reactor which is successfully stabilized by the proposed control strategy. Finally, the paper is concluded in Section 6.

2 Mathematical Preliminaries

2.1 Notation

Here $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, kernel and resolvent of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively. For a linear operator $A : \mathcal{D}(A) \subset X \to X$ and a fixed $s_0 \in \rho(A)$, define the scale spaces $X_1 := (\mathcal{D}(A), \|(s_0 - A) \cdot \|)$ and $X_{-1} = \overline{(X, \|(s_0 - A)^{-1} \cdot \|)}$ [31, Sec. 2.10]. The scale spaces are related by $X_1 \subset X \subset X_{-1}$ where the inclusions are dense and with continuous embeddings. The extension of A to X_{-1} is denoted by A_{-1}. The Λ-extension of an operator P is denoted by P_Λ (see (1)).

2.2 Regular Linear Systems

Consider a well-posed linear system (A, B, C, D), where $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup, $B \in \mathcal{L}(U, X_{-1})$ is the control operator, $C \in \mathcal{L}(X_1, Y)$ is the observation operator, and $D \in \mathcal{L}(U, Y)$. We assume that the spaces X, U, and Y are separable Hilbert spaces and that U and Y are finite-dimensional.

The operator B is called an *admissible input operator* for A if for some $\tau > 0$, the operator $\Phi_\tau \in \mathcal{L}(L^2(0,\infty;U), X_{-1})$ defined as [31, Sec. 4.2]:

$$\Phi_\tau u = \int_0^\tau T(\tau - s)Bu(s)ds,$$

satisfies $\mathcal{R}(\Phi_\tau) \subset X$. Correspondingly, the operator C is called an *admissible output operator* for A if for some $\tau > 0$, there exists a K_τ such that [31, Sec. 4.3]:

$$\int_0^\tau \|CT(s)x\|^2 ds \leq K_\tau \|x\|^2, \qquad \forall x \in \mathcal{D}(A).$$

Furthermore, if there exists a K such that $K_\tau \leq K$ for all $\tau > 0$, then C is called *infinite-time admissible*. The Λ-*extension* of the operator C is defined as [34]:

$$C_\Lambda x = \lim_{\lambda \to \infty} \lambda C(\lambda - A)^{-1}x, \tag{1}$$

and the domain of C_Λ consists of those elements $x \in X$ for which the limit exists.

Let G denote the transfer function of the system (A, B, C, D). The transfer function is called *regular* if $\lim_{\lambda\to\infty} G(\lambda)u = Du$ ($\lambda \in \mathbb{R}$) for all $u \in U$ [35, Thm. 1.3], in which case (A, B, C, D) is called a *regular linear system*.

The transfer function G of a regular system is given by:

$$G(s) := G_0(s) + D := C_\Lambda(s - A)^{-1}B + D,$$

and in the time domain the system is described by the following equations:

$$\dot{x}(\zeta, t) = Ax(\zeta, t) + Bu(t), \qquad x(\zeta, 0) = x_0(\zeta) \tag{2a}$$
$$y(t) = C_\Lambda x(\zeta, t) + Du(t). \tag{2b}$$

Throughout this paper, we assume that we are dealing with regular linear systems with admissible B and C.

2.3 Cayley-Tustin Time Discretization

Consider a system given in (2). Given a time discretization parameter $h > 0$, the Tustin time discretization of (2) is given by

$$\frac{x(jh) - x((j-1)h)}{h} \approx A\frac{x(jh) - x((j-1)h)}{2} + Bu(jh)$$
$$y(jh) \approx C\frac{x(jh) - x((j-1)h)}{2} + Du(jh)$$

for $j \geq 1$, where we omitted the spatial dependence of x for brevity. Let $u_j^{(h)}/\sqrt{h}$ be the approximation of $u(t)$ on the interval $t \in ((j-1)h, jh)$, e.g., by the mean value sampling used in [10]:

$$\frac{u_j^{(h)}}{\sqrt{h}} = \frac{1}{h}\int_{(j-1)h}^{jh} u(t)dt.$$

It has been shown in [10] that the Cayley-Tustin discretization is a convergent time discretization scheme for input-output stable system nodes satisfying $\dim U = \dim Y = 1$ in the sense that $y_j^{(h)}/\sqrt{h}$ converges to $y(t)$ in several different ways as $h \to 0$. The discussion in [10, Sec. 6] further implies that the same holds for any finite dimensional U and Y. Thus, writing $y_j^{(h)}/\sqrt{h}$ and $u_j^{(h)}/\sqrt{h}$ in place of $y(jh)$ and $u(jh)$, respectively, simple computations yield the *Cayley-Tustin discretization* of (2) as:

$$x(\zeta, k) = A_d x(\zeta, k-1) + B_d u(k), \quad x(\zeta, 0) = x_0(\zeta)$$
$$y(k) = C_d x(\zeta, k-1) + D_d u(k),$$

where:

$$\begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} := \begin{bmatrix} -I + 2\delta(\delta - A)^{-1} & \sqrt{2\delta}(\delta - A_{-1})^{-1}B \\ \sqrt{2\delta}C(\delta - A)^{-1} & G(\delta) \end{bmatrix}$$

and $\delta := 2/h$. Clearly one must have $\delta \in \rho(A)$, so that the resolvent operator is well-defined. Thus, for a large enough δ, the discretization can be applied to unstable systems as well.

Remark 1 *Due to the standing assumptions it is easy to see that the discretized operators are bounded. In fact, the boundedness of B_d and C_d already follows from $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, respectively, and for D_d being bounded it would suffice that the system* (2) *is well-posed rather than regular.*

3 Model Predictive Control

The moving horizon regulator is based on a similar formulation emerging from the finite-dimensional system theory (see e.g. [20]). A corresponding controller in the infinite-dimensional case is presented, e.g., in [39]. At a given sampling time k, the objective function with constraints is given by:

$$\begin{aligned}
\min_{u} & \sum_{j=k+1}^{\infty} \langle y_{k+j}, Q y_{k+j}\rangle_Y + \langle u_{k+j}, R u_{k+j}\rangle_U \\
\text{s.t.} \quad & x_j = A_d x_{j-1} + B_d u_j \\
& y_j = C_d x_{j-1} + D_d u_j \\
& u_{\min} \leq u_j \leq u_{\max} \\
& y_{\min} \leq y_j \leq y_{\max},
\end{aligned} \tag{3}$$

where Q and R are positive self-adjoint weights on the outputs y_j and inputs u_j, respectively. Here it is assumed for simplicity that U and Y are (finite-dimensional) real-valued spaces. For consideration of the MPC with complex input and output spaces, see [11], where the authors considered MPC for the Schrödinger equation.

The infinite-horizon objective function (3) can be cast into a finite-horizon objective function under certain assumptions on the inputs beyond the control horizon. Furthermore, a penalty term needs to be added to the objective function to account for the inputs and outputs beyond the horizon. We will present two approaches on this depending on the stability of the original plant.

3.1 Stable systems

If A is the generator of a (strongly) stable C_0-semigroup, we may assume that the input is zero beyond the control horizon N, i.e., $u_{k+N+i} = 0, \forall i \in \mathbb{N}$, and add a corresponding output penalty term. Under the assumption that C is infinite-time admissible for A, the terminal output penalty term can be written as a state penalty term, so that the finite-horizon objective function is given by:

$$\min_{u^N} \sum_{j=k+1}^{k+N} \langle y_j, Q y_j\rangle_Y + \langle u_j, R u_j\rangle_U + \langle x_{k+N}, \bar{Q} x_{k+N}\rangle_X \tag{4}$$

with the same constraints as in (3), and where N is the length of the control horizon.

The operator $\bar{Q}$ can be calculated from the positive self-adjoint solution of the following discrete-time Lyapunov equation:

$$A_d^* \bar{Q} A_d - \bar{Q} = -C_d^* Q C_d, \tag{5}$$

or equivalently (see e.g. [5, Ex. 4.30]) the continuous-time Lyapunov equation:

$$A^* \bar{Q} + \bar{Q} A = -C^* Q C \tag{6}$$

on the dual space of X_{-1}. The assumption of C being infinite-time admissible for A is required as it is equivalent to the continuous-time Lyapunov equation having solutions [31, Thm. 5.1.1]. Furthermore, as A is assumed to be stable, we have that the operator $\bar{Q} \in \mathcal{L}(X)$ given by:

$$\bar{Q}x = \lim_{\tau\to\infty} \int_0^{\tau} T^*(t) C^* Q C T(t) x dt, \quad \forall x \in \mathcal{D}(A), \tag{7}$$

is the unique positive self-adjoint solution of the continuous-time Lyapunov equation (6) (equivalently (5)).

Now that we have established that the finite-horizon objective function (4) is well-defined, to further manipulate the objective function (4) we introduce the notation $Y_k := (y_{k+n})_{n=1}^N \in Y^N$ and $U_k := (u_{k+n})_{n=1}^N \in U^N$. Hence, a manipulation of the objective function (4) leads to the following quadratic optimization problem:

$$\min_{U_k} \langle U_k, H U_k\rangle_{U^N} + 2\langle U_k, P x_k\rangle_{U^N} + \langle x_k, \bar{Q} x_k\rangle_X, \tag{8}$$

where $H \in \mathcal{L}(U^N)$ is positive and self-adjoint given by:

$$h_{i,j} = \begin{cases} D_d^* Q D_d + B_d^* \bar{Q} B_d + R, & \text{for } i = j \\ D_d^* Q C_d A_d^{i-j-1} B_d + B_d^* \bar{Q} A_d^{i-j} B_d, & \text{for } i > j \\ h_{j,i}^*, & \text{for } i < j \end{cases}$$

and $P \in \mathcal{L}(X, U^N)$ is given by $P = (D_d^* Q C_d A_d^{k-1} + B_d^* \bar{Q} A_d^k)_{k=1}^N$

The objective function (8) is subjected to constrains $U_{\min} \leq U_k \leq U_{\max}$ and $Y_{\min} \leq (SU_k + Tx_k) \leq Y_{\max}$ which can be written in the form:

$$\begin{bmatrix} I \\ -I \\ S \\ -S \end{bmatrix} U_k \leq \begin{bmatrix} U_{\max} \\ -U_{\min} \\ Y_{\max} - Tx_k \\ -Y_{\min} + Tx_k \end{bmatrix}, \tag{9}$$

where $S \in \mathcal{L}(U^N, Y^N)$ is given by:

$$s_{i,j} = \begin{cases} D_d, & \text{for } i = j \\ C_d A_d^{i-j-1} B_d, & \text{for } i > j \\ 0, & \text{for } i < j \end{cases}$$

and $T \in \mathcal{L}(X, Y^N)$ is given by $T = (C_d A_d^{k-1})_{k=1}^N$.

Considering a finite-dimensional output space $U = \mathbb{R}^m$, the inner products in the objective function given in (8) are simply vector products, and we have a finite dimensional quadratic optimization problem:

$$\min_{U_k} J(U_k, x_k) = U_k^T H U_k + 2 U_k^T (P x_k). \tag{10}$$

Note that the term $\langle x_k, \bar{Q}x_k \rangle_X$ can be neglected as x_k is the initial condition for step $k+1$ and cannot be affected by the control input. Furthermore, as all the operators related to the objective function and the linear constraints are bounded under the standing assumptions, the quadratic optimization problem is exactly of the same form as the ones obtained for finite-dimensional systems. Thus, we obtain the convergence and stability results for free by the MPC theory on finite-dimensional systems (see e.g. [28]). To highlight this observation, we present the following result:

Theorem 2 *Assume that A is the generator of a strongly stable C_0-semigroup and that C is an infinite-time admissible observation operator for A. Then, the input sequence (U_k) (and hence the sequence (u_k)) obtained as the solution of the feasible quadratic optimization problem* (10) *with constraints* (9) *converges to zero.*

PROOF. *By the preceding argumentation, the resulting MPC problem is equivalent to a finite-dimensional one, and thus, the result follows from standard finite-dimensional MPC theory.*

Remark 3 *Due to the assumed strong stability of the semigroup generated by A, the state of the system under the MPC control law goes asymptotically to zero for all initial states $x_0 \in \mathcal{D}(A)$ for which the problem is feasible as the control inputs decay to zero by Theorem 2.*

3.2 Exponentially stabilizable systems

Let us now assume that the pair (A, B) is exponentially stablizable, i.e., there exists an admissible feedback operator $K \in \mathcal{L}(X_1, U)$ such that $A + BK_\Lambda$ is the generator of an exponentially stable C_0-semigroup [36, Def. 3.1]. Optimal (in terms of minimizing the continuous version of (3)) state feedback operator is obtained using the maximal solution $\bar{R} \in \mathcal{L}(X)$ of the continuous-time Riccati equation [18, Def. 10.1.2] (see also [37]) :

$$K^*SK = A^*\bar{R} + \bar{R}A + C^*QC \tag{11}$$

on $\mathcal{D}(A)$, where $S := R + D^*QD$ and $K := -S^{-1}(B_\Lambda^*\bar{R} + D^*QC)$ yields the optimal feedback operator. Moreover, it follows from the proof of [4, Thm. 9] that the solutions of (11) are equivalent to the solutions of the discrete-time Riccati equation:

$$K_d^*S_dK_d = A_d^*\bar{R}A_d - \bar{R} + C_d^*QC_d, \tag{12}$$

where $S_d := B_d^*\bar{R}B_d + R + D_d^*QD_d$ and $K_d := -S_d^{-1}(A_d\bar{R}B_d + D_d^*QC_d)$ yields the optimal state feedback for the discrete-time system with the maximal $\bar{R}$. Furthermore, $A_d + B_dK_d$ corresponds to the Cayley-Tustin discretization of $A + BK_\Lambda$. Thus, as K is an exponentially stablizing feedback for (A, B), equivalently $A_{K_d} := A_d + B_dK_d$ is power stable.

Returning to the MPC problem, we assume that the optimal state feedback is utilized beyond the control horizon, i.e., $u_{k+N+i} = K_dx_{k+N+i-1}, \forall i \in \mathbb{N}$. Thus, the input and output terminal penalties can be expressed as state terminal penalties by solving the discrete-time Lyapunov equations:

$$\begin{aligned} A_{K_d}^*\bar{Q}_1A_{K_d} - \bar{Q}_1 &= -K_d^*RK_d \\ A_{K_d}^*\bar{Q}_2A_{K_d} - \bar{Q}_2 &= -(C_d + K_dD_d)^*QC_d + K_dD_d) \end{aligned}$$

or equivalently their continuous-time counterparts:

$$\begin{aligned} A_K^*\bar{Q}_1 + \bar{Q}_1A_K &= -K^*RK \\ A_K^*\bar{Q}_2 + \bar{Q}_2A_K &= -(C + DK)^*Q(C + DK), \end{aligned}$$

where $A_K := A + BK_\Lambda$. Note that as A_K is the generator of an exponentially stable semigroup and K and C are admissible for A_K by their admissibility for A and [31, Thm 5.4.2], the positive self-adjoint solutions of the Lyapunov equations are unique by [31, Thm. 5.1.1] and obtained similar to (7).

Finally, the input and output terminal penalties are given by $\langle x_{k+N}, \bar{Q}_1x_{k+N} \rangle$ and $\langle x_{k+N}, \bar{Q}_2x_{k+N} \rangle$, respectively. Thus, the quadratic formulation of the MPC problem is given as in the stable case, except that in H and P the operator $\bar{Q}$ must be replaced with $\bar{Q}_1 + \bar{Q}_2$.

Note that the full state feedback $u = Kx$ optimally solves the *unconstrained* minimization problem (3). Thus, in order to utilize it in the constrained setting, we need to first assume that the system is stabilizable by a sequence of inputs satisfying the input constraints. Under this assumption, MPC is utilized to steer the system into a region where $u_{\min} \leq Kx \leq u_{\max}$, at which point we can switch from MPC to the state feedback control. The existence of a constrained stabilizing input sequence can be guaranteed by allowing sufficiently high-gain inputs to cancel output the unstable dynamics of the system.

Theorem 4 *Assume that the system* (2) *is stabilizable by a sequence of inputs satisfying the input constraints. Then, the dual-mode control consisting of MPC and optimal state feedback optimally stabilizes the system while satisfying the input constraints.*

PROOF. *As the stabilization cost is included in the MPC problem, the optimal solutions of* (10) *asymptotically steer the state of the system towards zero. Once the state reaches the region where state feedback satisfies the input constraints, MPC can be switched to it to finalize stabilization.*

In practice, finding the optimal feedback K is rather challenging as the Riccati equation (11) can rarely be solved in analytic closed-form. Instead, some other stabilizing feedback can be used as a terminal penalty and stabilizing feedback as well. One possible option is to use output feedback $u_k = K_y y_k$. This is a valid choice as well as regularity of the system is preserved under output feedback (see [34]), and rather straightforward computations using Sherman-Morrison-Woodbury formula show that $A_d + B_d K_y (I - D_d K_y)^{-1} C_d$ corresponds to the Cayley-Tustin discretization of $A + BK_y(I - DK_y)^{-1}C$, i.e., A after output feedback. Apart from optimality, the result of Theorem 4 holds for any stabilizing feedback.

4 Wave Equation

As an example of a stable system, consider the wave equation on a 1-D spatial domain $\zeta \in [0, 1]$ with viscous damping at one end and boundary control u and boundary observation y at the other end given by:

$$\frac{\partial^2}{\partial t^2} w(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial}{\partial \zeta} w(\zeta, t) \right) \tag{13a}$$

$$0 = T(\zeta) \frac{\partial}{\partial \zeta} w(1, t) + \frac{\kappa}{\rho} \frac{\partial}{\partial t} w(1, t) \tag{13b}$$

$$u(t) = \frac{\partial}{\partial \zeta} w(0, t) \tag{13c}$$

$$y(t) = \frac{\partial}{\partial t} w(0, t), \tag{13d}$$

where $\kappa > 0$. For simplicity we assume that the mass density ρ and the Young's modulus T are constants. We further assume that $\kappa \neq \sqrt{\rho T}$, which will be needed in Section 4.3.

In order to write (13) in a more compact form, let us first define a new state variable $x = [x_1,\, x_2]^T := [\rho \partial_t w,\, \partial_\zeta w]^T$ with state space $X = L_2(0, 1; \mathbb{R}^2)$ and an auxiliary matrix operator $\mathcal{H}(\zeta) := \mathrm{diag}(\rho(\zeta)^{-1},\ T(\zeta))$. Now define the operator $\mathcal{A}$ by:

$$\mathcal{A}x(\zeta, t) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta) x(\zeta, t) \right)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ x \in X : \mathcal{H}x \in H^1(0, 1; \mathbb{R}^2),\ Tx_2(1) = -\tfrac{\kappa}{\rho} x_1(1) \right\},$$

so that the first two lines of (13) can be equivalently written as $\dot{x} = \mathcal{A}x$. Finally, by defining operators $\mathcal{B}$ and $\mathcal{C}$ as $\mathcal{B}x := Tx_2(0)$ and $\mathcal{C}x := \rho^{-1} x_1(0)$, the system (13) can be equivalently written as:

$$\dot{x}(t) = \mathcal{A}x(t) \tag{14a}$$

$$u(t) = \mathcal{B}x(t) \tag{14b}$$

$$y(t) = \mathcal{C}_\Lambda x(t), \tag{14c}$$

which corresponds to the port-Hamiltonian formulation of the wave equation (see, e.g., [12, Ex. 9.2.1]).

In order to further write the system (14) in the usual state-space form, define the operator A as the restriction of $\mathcal{A}$ to the kernel of $\mathcal{B}$, i.e., $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ with domain $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$. Due to the definitions of $\mathcal{A}$ and $\mathcal{B}$, it can be shown using [32, Thm. III.2] that A is the generator of an exponentially stable C_0-semigroup. Consequently, the double $(\mathcal{A}, \mathcal{B})$ is a boundary control system in the sense of [31, Def. 10.1.1]. Thus, by [31, Prop. 10.1.2, Rem. 10.1.4], there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that (14) can be equivalently written as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{15a}$$

$$y(t) = \mathcal{C}_\Lambda x(t) + Du(t) \tag{15b}$$

where $D := \lim_{s \to \infty} G_0(s)$ which is well-defined assuming that the system is regular [12, Def. 13.1.11]. Note that the transfer function of the system (15) is $G_0(s) = \mathcal{C}_\Lambda (s - A)^{-1} B$ as the feedthrough D is not present in the original boundary control system (14).

By [31, Rem. 10.1.5], the operator B can be found by solving the abstract elliptic problem $\mathcal{A}f = sf$, $\mathcal{B}f = u$ for any $u \in U$ and $s \in \rho(A)$, the unique solution of which satisfies $f = (s - A_{-1})^{-1} Bu$. Since here A is the generator of an exponentially stable C_0-semigroup, we can choose $s = 0$ and obtain the solution $f = (\rho/\kappa,\, -T^{-1})^T u$, and finally, the operator B is defined as:

$$Bu := A_{-1} \begin{bmatrix} \frac{\rho}{\kappa} \\ -\frac{1}{T} \end{bmatrix} u. \tag{16}$$

4.1 Discretized Operators

Assume that ρ and T are constants and consider the equation $\dot{x}(t) = Ax(t)$. Using the Laplace transform yields

$$sx(\zeta, s) - x(\zeta, 0) = \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} 0 & T \\ \rho^{-1} & 0 \end{bmatrix} x(\zeta, s) \right),$$

that is,

$$\frac{\partial}{\partial \zeta} x(\zeta, s) = \begin{bmatrix} 0 & \rho s \\ T^{-1} s & 0 \end{bmatrix} x(\zeta, s) - \begin{bmatrix} 0 & \rho \\ T^{-1} & 0 \end{bmatrix} x(\zeta, 0).$$

The above is an ordinary differential equation of the form:

$$\frac{\partial}{\partial \zeta} x(\zeta, s) = \overline{A} x(\zeta, s) - \overline{B} x(\zeta, 0),$$

the solution of which is given by:

$$x(\zeta,s) = e^{\overline{A}\zeta}x(0,s) - \int_0^\zeta e^{\overline{A}(\zeta-\eta)}\overline{B}x(\eta,0)d\eta \tag{17}$$

where:

$$e^{\overline{A}\zeta} = \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) & \sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \\ \left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) & \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \end{bmatrix}.$$

Recall that $\mathcal{D}(A)$ has the boundary conditions $Tx_2(1) + \frac{\kappa}{\rho}x_1(1) = 0$ and $Tx_2(0) = 0$, based on which $x(0,s)$ in (17) can be solved. Eventually, (17) is given by:

$$\begin{aligned} x(\zeta,s) = &\frac{\rho}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}s\right)} \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \\ \left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \end{bmatrix} \times \\ &\int_0^1 \left(\frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right) + \cosh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right)\right)x_1(\eta,0) \\ &\quad + \left(\kappa\cosh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right)\right)x_2(\eta,0)d\eta \\ &- \int_0^\zeta \begin{bmatrix} \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) & \rho\cosh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) \\ T^{-1}\cosh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) & \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) \end{bmatrix} x(\eta,0)d\eta \\ := &(s-A)^{-1}x(\zeta,0), \end{aligned}$$

which yields the expression for the resolvent operator, from which we also obtain the operator $A_d = -I + 2\delta(\delta - A)^{-1}$.

Based on the expression we derived for the operator B in (16), we have:

$$(\delta - A_{-1})^{-1}B = -\begin{bmatrix} \frac{\rho}{\kappa} \\ -\frac{1}{T} \end{bmatrix} + \delta(\delta - A_{-1})^{-1}\begin{bmatrix} \frac{\rho}{\kappa} \\ -\frac{1}{T} \end{bmatrix},$$

and a direct calculation yields that:

$$\begin{aligned} B_d = &\sqrt{\frac{\rho}{T}} \frac{\sqrt{2\delta}\left(\frac{\rho T}{\kappa} - \kappa\right)\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right)}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)} \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \\ \left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \end{bmatrix} \\ &+ \sqrt{2\delta}\begin{bmatrix} \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) - \frac{\rho}{\kappa}\cosh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \\ \frac{1}{T}\cosh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) - \frac{1}{\kappa}\sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\zeta\right) \end{bmatrix}, \end{aligned}$$

which can be further simplified using the properties of hyperbolic: functions to

$$\begin{aligned} B_d = &\frac{-\sqrt{2\delta}}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)} \times \\ &\begin{bmatrix} \rho\cosh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta-1)\right) + \kappa\sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta+1)\right) \\ \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta-1)\right) - \frac{\kappa}{T}\cosh\left(\sqrt{\frac{\rho}{T}}\delta(\zeta-1)\right) \end{bmatrix}. \end{aligned}$$

Furthermore, we obtain:

$$\begin{aligned} C_d x(\zeta) = &\frac{\sqrt{2\delta}}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right)\kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)} \times \\ &\int_0^1 \left(\frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\frac{\rho}{T}}\delta(1-\eta)\right) + \cosh\left(\sqrt{\frac{\rho}{T}}\delta(1-\eta)\right)\right)x_1(\eta) \\ &\quad + \left(\kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta(1-\eta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta(1-\eta)\right)\right)x_2(\eta)d\eta. \end{aligned}$$

Finally, based on the expression of B_d it is easy to see that the operator $D_d = G_0(\delta) = \mathcal{C}_\Lambda(\delta - A_{-1})^{-1}B$ is given by:

$$D_d = -\frac{1}{\sqrt{\rho T}} \frac{\kappa\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \sqrt{\rho T}\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)}. \tag{18}$$

We note that $\lim_{\delta\to\infty} G_0(\delta) = -(\rho T)^{-1/2}$ to verify that (14) indeed is a regular linear system.

4.2 Adjoint Operators

In order to find the adjoints of the discretized operators computed in the previous section, we equip the state-space X with the L^2 inner product, and the input and output spaces are equipped with the real scalar product. In order to find A_d^*, we find the adjoint of the resolvent operator $(s - A)^{-1}$:

$$\begin{aligned} &\langle(\delta - A)^{-1}x, z\rangle_X \\ = &\int_0^1 \frac{\rho z^*(\zeta)}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}s\right)} \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \\ \left(\sqrt{\rho T}\right)^{1}\sinh\left(\sqrt{\frac{\rho}{T}}s\zeta\right) \end{bmatrix} \times \\ &\int_0^1 \left(\frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right) + \cosh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right)\right)x_1(\eta) \\ &\quad + \left(\kappa\cosh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s(1-\eta)\right)\right)x_2(\eta)d\eta d\zeta \\ &- \int_0^1 z^*(\zeta)\int_0^\zeta \begin{bmatrix} \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) & \rho\cosh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) \\ T^{-1}\cosh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) & \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\zeta-\eta)\right) \end{bmatrix} x(\eta)d\eta d\zeta \\ = &\int_0^1\int_0^1 \rho\frac{z_1^*(\eta)\cosh\left(\sqrt{\frac{\rho}{T}}s\eta\right) + z_2^*(\eta)\left(\sqrt{\rho T}\right)^{-1}\sinh\left(\sqrt{\frac{\rho}{T}}s\eta\right)}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}s\right)}d\eta \times \\ &\begin{bmatrix} \frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(1-\zeta)\right) + \cosh\left(\sqrt{\frac{\rho}{T}}s(1-\zeta)\right) \\ \kappa\cosh\left(\sqrt{\frac{\rho}{T}}s(1-\zeta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}s(1-\zeta)\right) \end{bmatrix}^* x(\zeta)d\zeta \\ &- \int_0^1\int_\zeta^1 z^*(\eta)\begin{bmatrix} \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\eta-\zeta)\right) & \rho\cosh\left(\sqrt{\frac{\rho}{T}}s(\eta-\zeta)\right) \\ T^{-1}\cosh\left(\sqrt{\frac{\rho}{T}}s(\eta-\zeta)\right) & \sqrt{\frac{\rho}{T}}\sinh\left(\sqrt{\frac{\rho}{T}}s(\eta-\zeta)\right) \end{bmatrix} d\eta x(\zeta)d\zeta \\ = &\langle x, (s-A)^{-*}z\rangle_X, \end{aligned}$$

and now, A_d^* is given by $A_d^* = -I + 2\delta(\delta - A)^{-*}$.

For B_d we have $\langle B_d u, x\rangle_X = u\langle B_d, x\rangle_X = uB_d^*x$, and in a similar manner, we obtain for C_d that:

$$
\begin{aligned}
yC_dx = &\frac{y\sqrt{2\delta}}{\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\delta\right) + \kappa\cosh\left(\sqrt{\frac{\rho}{T}}\delta\right)}\times \\
&\int_0^1 \left(\frac{\kappa}{\sqrt{\rho T}}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right) + \cosh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)\right) x_1(\eta,0) \\
&\quad + \left(\kappa\cosh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right) + \sqrt{\rho T}\sinh\left(\sqrt{\tfrac{\rho}{T}}\delta(1-\eta)\right)\right) x_2(\eta,0)d\eta. \\
&= \langle C_d^* y, x\rangle_X.
\end{aligned}
$$

Finally, D_d is self-adjoint.

4.3 Solution of the Lyapunov equation

In this section, we derive the positive solution for the continuous Lyapunov equation (6), which is realized by utilizing the spectral representation of A. Let us at first find the eigenvalues and eigenvectors of the operator A. A direct computation shows that the solution of the eigenvalue equation $A\phi_k = \lambda_k\phi_k$ is of the form:

$$
\begin{aligned}
\phi_{1,k}(\zeta) &= \alpha\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right) + \beta\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right) \\
\phi_{2,k}(\zeta) &= \frac{\alpha}{\sqrt{\rho T}}\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right) - \frac{\beta}{\sqrt{\rho T}}\exp\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right).
\end{aligned}
$$

Since $\phi_k \in \mathcal{D}(\mathcal{A})$, we must have $\phi_{2,k}(0) = 0$, which yields $\alpha = \beta$. Thus, the eigenvectors of A are of the form:

$$
\phi_k(\zeta) = \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right) \\ \frac{1}{\sqrt{\rho T}}\sinh\left(\sqrt{\frac{\rho}{T}}\lambda_k\zeta\right) \end{bmatrix},
$$

and the eigenvalues λ_k are determined from the condition $T\phi_{2,k}(1) = -\frac{\kappa}{\rho}\phi_{1,k}(1)$, i.e.,

$$
\sqrt{\frac{T}{\rho}}\sinh\left(\sqrt{\frac{\rho}{T}}\lambda_k\right) + \frac{\kappa}{\rho}\cosh\left(\sqrt{\frac{\rho}{T}}\lambda_k\right) = 0.
$$

Using the exponential form of the hyperbolic functions we obtain that one of the eigenvalues is given by:

$$
\lambda_0 = \frac{1}{2}\sqrt{\frac{T}{\rho}}\log\left(\frac{\sqrt{\rho T}-\kappa}{\sqrt{\rho T}+\kappa}\right), \tag{19}
$$

which is real if $\kappa < \sqrt{\rho T}$. Finally, by the periodicity of the exponential function along the imaginary axis, we obtain that in general the eigenvalues are given by $\lambda_k = \lambda_0 + \sqrt{T/\rho}k\pi i$ for $k \in \mathbb{Z}$.

We note that damped wave equations have been considered, e.g., in [3] and [38, Sect. 4] - both referring to the original work by Rideau [27] - where similar spectra were obtained. Furthermore, it can be seen from (19) that the assumption $\kappa \neq \sqrt{\rho T}$ is required to ensure $\sigma(A) \neq \emptyset$, which is further required by [3, Thm. 3.5] to ensure that the eigenvectors of A constitute a Riesz basis for X. Indeed, we can define an invertible operator:

$$
M := \begin{bmatrix} \cosh\left(\sqrt{\frac{\rho}{T}}\lambda_0\zeta\right) & -\sqrt{\rho T}\sinh\left(\sqrt{\frac{\rho}{T}}\lambda_0\zeta\right) \\ i\sinh\left(\sqrt{\frac{\rho}{T}}\lambda_0\zeta\right) & -i\sqrt{\rho T}\cosh\left(\sqrt{\frac{\rho}{T}}\lambda_0\zeta\right) \end{bmatrix},
$$

so that

$$
M\phi_k = \begin{bmatrix} \cos(k\pi\zeta) \\ \sin(k\pi\zeta) \end{bmatrix}
$$

is an orthonormal basis in X, and the biorthogonal sequence [31, Def. 2.5.1] $(\bar{\phi}_k)$ to (ϕ_k) is given by $\bar{\phi}_k = M^*M\phi_k$.

Let us now return to the Lyapunov equation and apply it to an arbitrary $x \in \mathcal{D}(A)$:

$$
A^*\bar{Q}x + \bar{Q}Ax + \mathcal{C}^*Q\mathcal{C}x = 0.
$$

By [31, Prop. 2.5.2], we can write every $x \in X$ as:

$$
x = \sum_{k\in\mathbb{Z}} \left\langle z, \bar{\phi}_k\right\rangle \phi_k,
$$

which yields:

$$
\sum_{k\in\mathbb{Z}} \left(A^*\bar{Q}\left\langle x, \bar{\phi}_k\right\rangle\phi_k + \bar{Q}A\left\langle x, \bar{\phi}_k\right\rangle\phi_k + \mathcal{C}^*Q\mathcal{C}\left\langle x, \bar{\phi}_k\right\rangle\phi_k\right) = 0,
$$

which by utilizing [31, Prop. 2.6.3] further yields:

$$
\sum_{k\in\mathbb{Z}} \left((A^*+\lambda_k)\bar{Q}\left\langle x, \bar{\phi}_k\right\rangle\phi_k + \mathcal{C}^*Q\mathcal{C}\left\langle x, \bar{\phi}_k\right\rangle\phi_k\right) = 0.
$$

The above especially holds if $(A^*+\lambda_k)\bar{Q}\left\langle x, \bar{\phi}_k\right\rangle\phi_k = -\mathcal{C}^*Q\mathcal{C}\left\langle x, \bar{\phi}_k\right\rangle\phi_k$ for all $k \in \mathbb{Z}$. Thus, for an arbitrary $k \in \mathbb{Z}$, we obtain:

$$
\bar{Q}\left\langle x, \bar{\phi}_k\right\rangle\phi_k = (-\lambda_k - A^*)^{-1}\mathcal{C}^*Q\mathcal{C}\left\langle x, \bar{\phi}_k\right\rangle\phi_k.
$$

As A is densely defined and $-\bar{\lambda}_k \in \rho(A)$ since $\lambda_k \in \sigma(A)$, we have by [31, Prop. 2.8.4] that $(-\lambda_k - A^*)^{-1} = \left((-\bar{\lambda}_k - A)^{-1}\right)^*$, so we obtain :

$$
\begin{aligned}
\bar{Q}\left\langle x, \bar{\phi}_k\right\rangle\phi_k &= \left((-\bar{\lambda}_k - A)^{-1}\right)^*\mathcal{C}^*Q\mathcal{C}\left\langle x, \bar{\phi}_k\right\rangle\phi_k \\
&= \left(\mathcal{C}(-\bar{\lambda}_k - A)^{-1}\right)^*Q\mathcal{C}\left\langle x, \bar{\phi}_k\right\rangle\phi_k.
\end{aligned}
$$

Finally, summation over $k \in \mathbb{Z}$ yields the solution:

$$
\bar{Q}x = \sum_{k\in\mathbb{Z}} \left\langle x, \bar{\phi}_k\right\rangle \left(\mathcal{C}(-\bar{\lambda}_k - A)^{-1}\right)^* Q\mathcal{C}\phi_k. \tag{20}
$$

Note that as $\mathcal{C}\phi_k = 1$ and $\mathcal{C}(-\lambda_k^* - A)^{-1}$ is uniformly bounded for all $k \in \mathbb{Z}$, the series in (20) is convergent (as it should since $\bar{Q} \in \mathcal{L}(X)$). Thus, for any $x \in X$ we may approximate:

$$\bar{Q}x \approx \bar{Q}_M x := \sum_{k=-M}^{M} \langle x, \bar{\phi}_k \rangle \left(\mathcal{C}(-\bar{\lambda}_k - A)^{-1}\right)^* Q\mathcal{C}\phi_k,$$

and it holds that $\lim_{M\to\infty} \|\bar{Q}x - \bar{Q}_M x\| = 0$, by which we can evaluate (20) to an arbitrary precision $\epsilon > 0$ by choosing a sufficiently large M. A suitable value for M can determined, e.g., by numerical experiments.

4.4 Simulation results for the wave equation

Consider the wave equation (13) with the parameter choices $\rho = T = 1$ and $\kappa = 0.75$. For the MPC, choose the optimization horizon as $N = 15$ and choose the input and output weights as $R = 10$ and $Q = 0.5$, respectively. For the Cayley-Tustin discretization, choose $h = 0.075$ so that $\delta \approx 26.67$. For numerical integration, an adaptive approximation of $d\zeta$ is used with 519 nodal points. To approximate the solution of the Lyapunov equation (20), we choose $M = 100$. The initial conditions for the wave equation in the port-Hamiltonian framework are given by $\partial_t w(\zeta) = \cos(\pi\zeta)$ and $\partial_\zeta w(\zeta) = \sin(\frac{1}{2}\pi\zeta)$.

The input and output constraints $-0.05 \leq u_k \leq 0.05$ and $-0.025 \leq y_k \leq 0.3$ are displayed in Figure 1 along with the control inputs $u(k)$ obtained from the MPC problem. The outputs of the system under the MPC and under no control are displayed as well. It can be seen that the MPC makes the output decay slightly faster in the beginning. Then control is imposed to satisfy the output constraints while the uncontrolled output violates them. Finally, a minor stabilizing control effort is imposed before both the MPC input and the output decay to zero. Naturally the uncontrolled output decays to zero as well due to the exponential stability of the considered system.

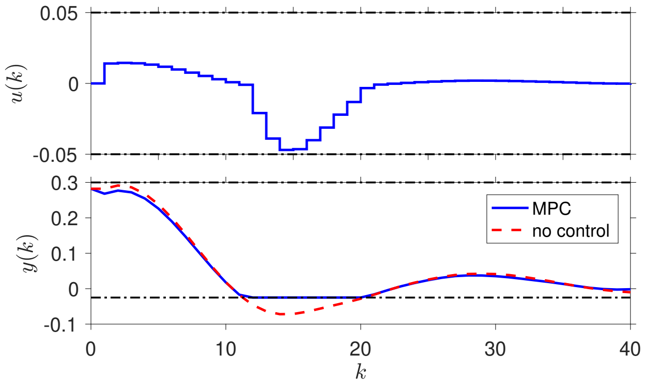

Figure 1. Above: MPC inputs $u(k)$ and the input constraints. Below: MPC and uncontrolled outputs and the output constraints.

Figure 2 displays the velocity profiles of the system under the model predictive control law and without control. No substantial differences can be observed in the velocity profiles, which is rather expected as the outputs in Figure 1 were rather close to one another. Relatively small differences in the outputs are natural as well, since the control inputs were constrained to rather small gain.

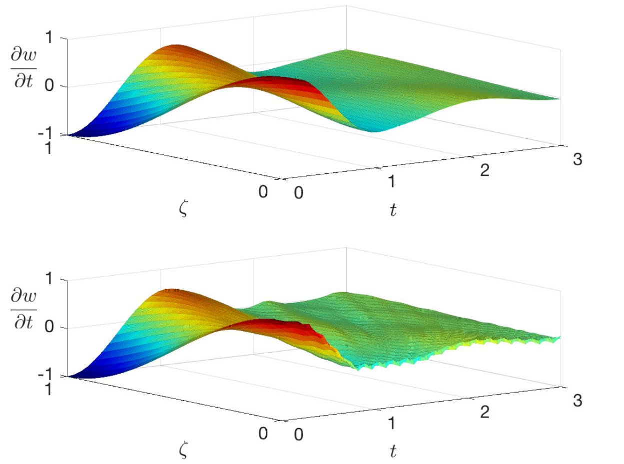

Figure 2. Above: the velocity profile of the wave equation without control. Below: the velocity profile under the model predicting control law.

5 Tubular reactor with recycle

As an example of an unstable system, consider a tubular reactor with recycle given as:

$$\frac{\partial}{\partial t}x(\zeta,t) = -v\frac{\partial}{\partial \zeta}x(\zeta,t) + \alpha x(\zeta,t) \tag{21a}$$
$$x(0,t) = rx(1,t) + (r-1)u(t) \tag{21b}$$
$$y(t) = x(1,t) \tag{21c}$$

on $\zeta \in [0,1]$, where the parameters are chosen as $v = 1, \alpha = 1/2$ and $r = 1/3$ so that the system has its spectrum in the right half plane but is exponentially stabilizable, e.g., by output feedback $u(t) = -y(t)$. Under this feedback, (21b) changes to $x(0,t) = (2r-1)x(1,t)$ but otherwise the system remains the same.

Similar to the wave equation in Section 4, we can compute the resolvent operator and find the discretized operator (A_d, B_d, C_d, D_d) and their adjoints. Since output feedback is used as a stabilizing terminal cost and in this case $D = 0$, for the terminal penalty one needs to solve the Lyapunov equation $A_s^*\bar{Q} + \bar{Q}A_s = -\mathcal{C}^*(Q+R)\mathcal{C}$, where A_s is the generator of the exponentially stable C_0-semigroup corresponding to the boundary control system (21) under output feedback $u(t) = -y(t)$. This can be done as in Section 4.3, except that the normalized eigenvectors of A_s already form an orthonormal basis in $X = L^2(0,1;\mathbb{R})$.

For the MPC problem formulation, the weights are chosen as $Q = 2$ and $R = 10$, and the input constraints are given by $-0.15 \leq u_k \leq 0.05$ while no output constraints are imposed. The optimization horizon is chosen as $N = 10$, and for approximation of the solution of the Lyapunov equation, 201 eigenvectors of A_s are used. For the Cayley-Tustin discretization, we choose $h = 0.1$ so that $\delta = 20$. The initial condition is given by $x_0(\zeta) = \frac{1}{2}\sin(\pi\zeta)$. For numerical integration, an adaptive approximation of $d\zeta$ is used with 510 nodal points.

In Figure 3, the dual-mode inputs and the outputs of the system under the dual-mode control are presented. For comparison, the output feedback control and the output under the feedback control are also presented. It can be seen that while the output feedback stabilizes the system faster, it does not satisfy the input constraints early on in the simulation. In the dual-model control, the MPC inputs first steer the output close to zero while satisfying the input constraints, and then at $k = 80$ it is switched to output feedback $u = -y$ which completes the stabilization.

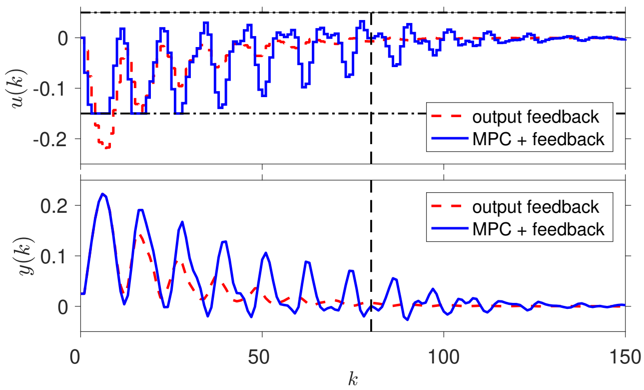

Figure 3. Above: dual-mode inputs, the input constraints and the output feedback. Below: outputs of the system under the dual-mode control and output feedback.

In Figure 4, the state profiles of the tubular reactor are displayed under the dual-mode and the feedback controls. The states behave according to what could be expected based on the outputs, that is, both states decay asymptotically to zero and the state under output feedback decays faster.

6 Conclusions

In this work, a linear model predictive controller for regular linear systems was designed, and it was shown that for stable systems, stability of the zero output regulator follows from the finite-dimensional MPC theory. For stabilizable systems, constrained stabilization was achieved by dual-mode control consisting of MPC and stabilizing feedback. The MPC design was demonstrated on an illustrative example where it was implemented for the boundary controlled wave equation. Constrained stabilization was demonstrated on a tubular reactor which had solely unstable eigenvalues. The performances of the

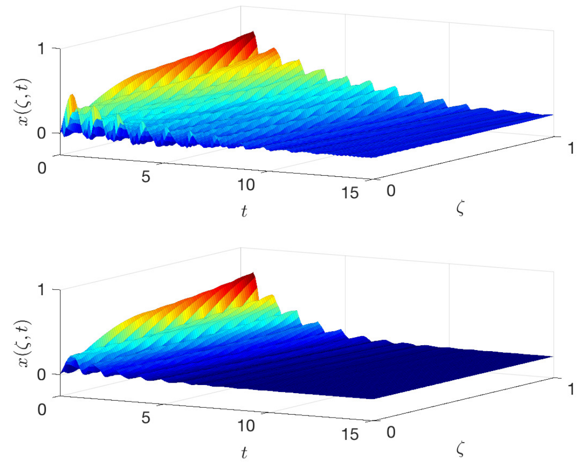

Figure 4. Above: the state profile of the tubular reactor under the-dual mode control. Below: the state profile under the output feedback.

control strategies were illustrated with numerical simulations.

It should be noted that the assumption of regularity was not in fact needed at any point when considering stable systems, but it was merely done for the convenience of the state-space presentation of the systems. Thus, the result of Theorem 2 can equivalently be formulated for well-posed instead of regular linear systems. Furthermore, by the obtained stability result, tracking of constant reference signals could be incorporated for MPC of regular linear systems by the classical MPC theory of finite-dimensional systems (see [24]). The result of Theorem 4 could be extended to well-posed linear systems as well, although state feedback stabilization and Riccati equations are much more involved concepts for these systems (see [18, 19]).

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