The internal model principle for boundary control systems with polynomially bounded exogenous signals

The Internal Model Principle for Boundary Control Systems with Polynomially Bounded Exogenous Signals ⋆

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Abstract: We extend the internal model principle for boundary control system to cover robust tracking of sinusoidal reference signals with polynomial coefficients. The internal model principle is presented in the form of both the internal model structure and the $\mathcal{G}$-conditions. A controller structure will be presented and its internal model properties will be analyzed in order to solve the tracking problem in a robust manner. As an example, a robust controller is constructed for the one-dimensional heat equation with Dirichlet boundary control at one endpoint and temperature measurement at the other endpoint of the interval. The performance of the controller is demonstrated by numerical simulations.

Keywords: robust control, output regulation, distributed parameter systems, regulator theory, feedback control, linear control systems

1. INTRODUCTION

The *internal model principle* comes into play when constructing controllers for *robust output regulation.* The goal of robust output regulation is to design a control law such that the output $y(t)$ of a given plant asymptotically follows some reference trajectory $y_{ref}(t)$ in spite of external disturbance signals $w(t)$ and/or perturbations and uncertainties in the parameters of the plant. One possibility of designing such a control law is to construct a dynamical error feedback controller to produce the required controls.

The internal model principle indicates that a controller can solve the robust output regulation problem if and only if it contains a sufficient *internal model* of the dynamics of the exosystem that generates the disturbance and reference signals. The internal model principle was first introduced in the context of finite-dimensional systems by Francis and Wonham (1975, 1976). Since then, the principle has been extended to infinite-dimensional systems, e.g., in Rebarber and Weiss (2003); Immonen (2007); Hämäläinen and Pohjolainen (2010); Paunonen and Pohjolainen (2010). Most recently, the principle has been generalized to regular linear systems in Paunonen and Pohjolainen (2014) and to boundary control systems in Humaloja and Paunonen (2018); Humaloja et al. (2018), where the considered signals where linear combinations of sinusoidal signals with constant coefficients.

As the main contribution of this paper, we extend the internal model principle to boundary control systems for robust tracking of sinusoidal signals with polynomial coefficients (see (2) for the exact signal structure). The extended internal model principle will be presented in the form of the $\mathcal{G}$*-conditions* as in (Paunonen and Pohjolainen, 2010, Def. 5.1), and as a side product also in the form of the *internal model structure* as in (Immonen and Pohjolainen, 2006, Def. 3.1). In fact, the result concerning the internal model structure is the main novel contribution, and thereafter the result on the $\mathcal{G}$-conditions can be derived by combining the results of Paunonen and Pohjolainen (2010, 2014) with some care in the technical details. We will also consider a controller structure from Hämäläinen and Pohjolainen (2002) and show that it satisfies the $\mathcal{G}$-conditions.

The structure of the paper is as follows. In Section 2, we present the plant, the class of considered reference signals and the controller. In Section 3, we present the robust output regulation problem and the internal model principle in the form of the internal model structure and the $\mathcal{G}$-conditions in Section 3.1. In Section 3.2, we review the controller structure from Hämäläinen and Pohjolainen (2002) to solve the robust output regulation problem. In Section 4, a numerical example is presented on tracking of a ramp signal for a one-dimensional heat equation with non-collocated Dirichlet boundary control and observation. Finally, the paper is concluded in Section 5.

Here $\mathcal{L}(X,Y)$ denotes the family of bounded operators from the normed space X to the normed space Y. The domain, range, kernel, resolvent and spectrum of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\rho(A)$ and $\sigma(A)$, respectively.

2. THE PLANT, EXOSYSTEM AND CONTROLLER

We consider plants that are boundary control systems in the sense of (Curtain and Zwart, 1995, Def. 3.3.2) given by the following equations.

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$$\dot{x}(t) = \mathcal{A}x(t), \qquad x(0) = x_0 \tag{1a}$$
$$u(t) + w(t) = \mathcal{B}x(t) \tag{1b}$$
$$y(t) = \mathcal{C}x(t) \tag{1c}$$

where the pair $(\mathcal{A}, \mathcal{B})$ is such that the operator $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ is the generator of a C_0-semigroup, and there exists a right inverse $B \in \mathcal{L}(U, X)$ of $\mathcal{B}$ such that $\mathcal{R}(B) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{B}Bu = u$ for all $u \in U$.[1] Furthermore, the operator $\mathcal{C}$ is such that $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$ and $\mathcal{C}B \in \mathcal{L}(U, Y)$. The state space X, the input space U, and the output space Y are assumed to be Hilbert spaces. Furthermore, we assume that the operator A is the generator of an exponentially stable C_0-semigroup.

The disturbance signal $w(t)$ and the reference signal $y_{ref}(t)$ to be tracked are assumed to be of the form

$$y_{ref}(t) = a_0(t) + \sum_{k=1}^{q} (a_k(t)\cos(\omega_k t) + b_k(t)\sin(\omega_k t)) \tag{2a}$$

$$w(t) = c_0(t) + \sum_{k=1}^{q} (c_k(t)\cos(\omega_k t) + d_k(t)\sin(\omega_k t)) \tag{2b}$$

where the coefficients $a_k(t), b_k(t), c_k(t), d_k(t)$ are polynomials of known degree but possibly with unknown coefficients, and the frequencies $\{\omega_k\}_{k=0}^q \subset \mathbb{R}$ are known and such that $0 = \omega_0 < \omega_1 < \ldots < \omega_q$. The plant could be subjected to other disturbances as well, e.g., more general boundary disturbances, distributed disturbances or input disturbances, but here we will restrict to the input disturbance case in order to reduce notational complexity.

Reference and disturbance signals like the ones in (2) can be generated by an *exosystem* of the form

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \tag{3a}$$
$$w(t) = Ev(t) \tag{3b}$$
$$y_{ref}(t) = -Fv(t) \tag{3c}$$

on a finite-dimensional space W, where S is such that $\sigma(S) = \{\omega_0, \pm i\omega_1, \ldots, \pm i\omega_q\}$, and $E \in \mathcal{L}(W, U)$ and $F \in \mathcal{L}(W, Y)$ corresponding to the polynomial coefficients $c_k(t), d_k(t)$ and $a_k(t), b_k(t)$ in $w(t)$ and $y_{ref}(t)$, respectively. The algebraic multiplicity associated with an eigenvalue $i\omega_k$ is denoted by n_k so that at least one of the corresponding polynomial coefficients $a_k(t), b_k(t), c_k(t), d_k(t)$ is of degree $n_k - 1$. Thus, the exosystem state space is of dimension $\dim W = n_0 + 2\sum_{k=1}^q n_k$. In the following, we will denote $-i\omega_k$ by $i\omega_{-k}$.

It should be noted that the exosystem is merely a theoretical tool for deriving the internal model results in the next section. In order to construct a robust controller, one does not need any information on the parameters E and F (due to robustness) but only on the eigenvalues of S and their algebraic multiplicities, which can be determined directly from the reference and disturbance signals.

The controller to be constructed to solve the robust output regulation problem is a dynamic error feedback controller of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \qquad z(0) = z_0 \tag{4a}$$
$$u(t) = Kz(t) \tag{4b}$$

on a Banach space Z, where $y(t) - y_{ref}(t) := e(t)$ is the regulation error. The controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are chosen such that robust output regulation is achieved, and they satisfy $\mathcal{G}_1 \in \mathcal{L}(Z)$, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$.

When the plant, the (virtual) exosystem and the controller are connected, i.e., $u(t), y(t)$ and $y_{ref}(t)$ are set equal in (1)–(4), the resulting closed-loop system can be written in the state space form on an extended state space $X_e := X \times Z$. The closed-loop system is given by

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0} \tag{5a}$$
$$e(t) = \mathcal{C}_e x_e(t) + D_e v(t) \tag{5b}$$

where the regulation error $e(t) = y(t) - y_{ref}(t)$ is chosen as the output, and the operators are given by

$$A_e := \begin{bmatrix} A - BK\mathcal{G}_2\mathcal{C} & \mathcal{A}BK - BK(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}BK) \\ \mathcal{G}_2\mathcal{C} & \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}BK \end{bmatrix},$$
$$B_e := \begin{bmatrix} \mathcal{A}BE - BES - BK\mathcal{G}_2(\mathcal{C}BE + F) \\ \mathcal{G}_2(\mathcal{C}BE + F) \end{bmatrix},$$
$$\mathcal{C}_e := [\mathcal{C} \;\; \mathcal{C}BK],$$

and $D_e := \mathcal{C}BE + F$ (see (Humaloja et al., 2018, Sec. III) for the derivation and details). Even though we restrict to the input disturbance case in our theoretical considerations, as an example in Section 4, we consider a case where the disturbance acts on a different part of the boundary than the control input, which corresponds to the more general formulation of Humaloja et al. (2018).

As shown in (Humaloja et al., 2018, Thm. III.1), if the observation operator $\mathcal{C}$ is admissible (Tucsnak and Weiss, 2009, Def. 4.3.1) for the C_0-semigroup generated by A, then A_e is the generator of a the C_0-semigroup $(T_e(t))_{t\geq 0}$ and $\mathcal{C}_e$ is admissible for $(T_e(t))_{t\geq 0}$. In particular, the closed-loop system is a *regular linear system* in the sense of Weiss (1994).

3. ROBUST OUTPUT REGULATION

The goal of output regulation is to design a control law such that the output of a given system follows a given reference trajectory. In order to discuss robustness, consider the class $\mathcal{O}$ of admissible perturbed systems defined as follows.

Definition 1. The operators $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F})$ belong to the class $\mathcal{O}$ of admissible perturbed systems if

(i) The triple $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is a boundary control system.
(ii) The observation operator $\tilde{\mathcal{C}}$ is admissible for the semigroup generated by $\tilde{A} := \tilde{\mathcal{A}}|_{\mathcal{N}(\tilde{\mathcal{B}})}$.
(iii) The values $\pm i\omega_k$ are in the resolvent set of $\tilde{A}$, i.e., $\{i\omega_k\}_{k=-q}^q = \sigma(S) \subset \rho(\tilde{A})$.
(iv) $\tilde{E} \in \mathcal{L}(W, U)$ and $\tilde{F} \in \mathcal{L}(W, Y)$.

We denote a right inverse of $\tilde{\mathcal{B}}$ by $\tilde{B}$. Note that for systems in class $\mathcal{O}$, the resulting closed-loop system $(\tilde{A}_e, \tilde{B}_e, \tilde{C}_e, \tilde{D}_e)$ is still regular. Further note that the last item in the preceding list translates to the structure of the signals $y_{ref}(t)$ and $w(t)$ as: $a_k(t), b_k(t), c_k(t), d_k(t)$ are

[1] Note that the right inverse B is different from the actual control operator that would appear if the the plant (1) was written in the state-space form. However, the formulation of (Curtain and Zwart, 1995, Def. 3.3.2) only requires knowledge of the right inverse which we here denote by B.

arbitrary polynomials of degree $\leq n_k - 1$ for all $k \in \{0, 1, \ldots, q\}$.

Now that we have presented the class $\mathcal{O}$ of perturbed systems, the robust output regulation problem can be presented as follows.

Problem 1. (The Robust Output Regulation Problem) For a given plant, choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that

1) the closed-loop system generated by A_e is exponentially stable,
2) for all initial states $x_{e0} \in X_e$ the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2(0, \infty; Y)$ for some $\alpha > 0$ independent of $x_{e0} \in X_e$,
3) if the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $x_{e0} \in X_e$ the regulation error satisfies $e^{\tilde{\alpha} \cdot} e(\cdot) \in L^2(0, \infty; Y)$ for some $\tilde{\alpha} > 0$ independent of $x_{e0} \in X_e$.

The following auxiliary results yields the equivalence between a controller solving the (robust) output regulation problem and the *regulator equations*

$$\Sigma S = A_e \Sigma + B_e \tag{6a}$$

$$0 = \mathcal{C}_e \Sigma + D_e \tag{6b}$$

having a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e)$. The result follows from (Paunonen and Pohjolainen, 2014, Thm. 4.1) and has been presented in (Humaloja et al., 2018, Thm. IV.3).

Theorem 1. Assume that the closed-loop system is regular and exponentially stablized by the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$. Then the controller solves the output regulation problem if and only if the regulator equations (6) have a solution Σ. The solution is unique when it exists.

Note that the preceding result can be applied to robust output regulation as well by requiring that the regulator equations (6) have a solution for all perturbed systems $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ for which the closed-loop system is exponentially stable.

3.1 The Internal Model Principle

In the following, we will give the first characterization of the internal model principle in terms of the *internal model structure* (Immonen and Pohjolainen, 2006, Def. 3.1).

Definition 2. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has the *internal model structure* if

$$\forall \Gamma, \Delta : \qquad \Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \quad \Rightarrow \quad \Delta = 0, \tag{7}$$

where $\Gamma \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$.

The following result shows that a controller having the internal model structure is equivalent to the controller solving the robust output regulation problem. Throughout the proof we will utilize the corollary of Theorem 1 that robust output regulation is achieved if and only if, for perturbed systems in class $\mathcal{O}$ that yield an exponentially stable closed-loop system, the (unique) solution Σ of (6a) also satisfies (6b).

Theorem 2. Assume that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ exponentially stabilizes the closed-loop system. Then, the controller solves the robust output regulation problem if and only if has the internal model structure.

Proof. Assume first that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the robust output regulation problem. Let $\Gamma \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ be such that $\Gamma S = \mathcal{G}_1 \Gamma + \Delta$. Leave the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ unperturbed and choose such perturbations of E and F that $\tilde{E} = -K\Gamma$ and $\tilde{F} = \Delta$. Now the operator $\Sigma = [0 \quad \Gamma]^T \in \mathcal{L}(W, X_e)$ satisfies $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e)$, and by using $\Gamma S = \mathcal{G}_1 \Gamma + \Delta$, we obtain that

$$\Sigma S = \begin{bmatrix} 0 \\ \Gamma S \end{bmatrix} = \begin{bmatrix} BK\left(\Gamma S - \mathcal{G}_1 \Gamma - \mathcal{G}_2 \Delta\right) \\ \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta \end{bmatrix} = A_e \Sigma + \tilde{B}_e,$$

i.e., $\Sigma S = A_e \Sigma + \tilde{B}_e$. Moreover, as a consequence of Theorem 1 we obtain that

$$0 = \mathcal{C}_e \Sigma + \tilde{D}_e = \mathcal{C}BK\Gamma - \mathcal{C}BK\Gamma + \Delta = \Delta$$

which implies that the controller has the internal model structure.

Now assume that the controller has the internal model structure and consider an arbitrary perturbed system $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$. Due to the assumed exponential stability of the closed-loop system, the first regulator equation $\Sigma S = \tilde{A}_e \Sigma + \tilde{B}_e$ has a unique solution $\Sigma = [\Pi \quad \Gamma]^T$ by Phóng (1991). The second line of the equation yields

$$\begin{aligned} \Gamma S &= \mathcal{G}_2 \tilde{\mathcal{C}} \Pi + (\mathcal{G}_1 + \mathcal{G}_2 \tilde{\mathcal{C}} \tilde{B} K)\Gamma + \mathcal{G}_2(\tilde{\mathcal{C}} \tilde{B} \tilde{E} + \tilde{F}) \\ &= \mathcal{G}_1 \Gamma + \mathcal{G}_2(\tilde{\mathcal{C}} \Pi + \tilde{\mathcal{C}} \tilde{B} K \Gamma + \tilde{\mathcal{C}} \tilde{B} \tilde{E} + \tilde{F}) \\ &= \mathcal{G}_1 \Gamma + \mathcal{G}_2(\tilde{\mathcal{C}}_e \Sigma + \tilde{D}_e), \end{aligned}$$

where $\tilde{\mathcal{C}}_e \Sigma + \tilde{D}_e \in \mathcal{L}(W, Y)$ as $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e) \subset \mathcal{D}(\tilde{\mathcal{C}}_e)$. Thus, by the internal model structure, $\tilde{\mathcal{C}}_e \Sigma + \tilde{D}_e = 0$, i.e., the controller solves the robust output regulation problem.

While the previous result yields the internal model principle in a rather compact form, it is not the most practical one in terms of constructing a robust regulating controller — mainly due to the fact that it is not straightforward to see how the controller parameters should be chosen in such a way that the controller would have the internal model structure. Instead, the internal model principle can be equivalently expressed in terms of the $\mathcal{G}$*-conditions* (Paunonen and Pohjolainen, 2010, Def. 5.1) as follows.

Definition 3. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$*-conditions* if

$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{8a}$$

and

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \tag{8b}$$

$$\mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k - 1} \subset \mathcal{R}(i\omega_k - \mathcal{G}_1) \tag{8c}$$

for all $k \in \{-q, \ldots, q\}$.

The following result proves the equivalence between the internal model structure and the $\mathcal{G}$-conditions, and more importantly, that a controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions.

Theorem 3. Assume that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ exponentially stabilizes the closed-loop system. Then, the controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions.

Proof. Assume first that the controller satisfies the $\mathcal{G}$-conditions. It has been shown in (Paunonen and Pohjolainen, 2010, Lem. 5.6) that if the controller satisfies the $\mathcal{G}$-conditions, then it has the internal model structure. Thus, by Theorem 2, the controller solves the robust output regulation problem. Note that even though the operator C_e is bounded in Paunonen and Pohjolainen (2010), neither the internal model structure nor the $\mathcal{G}$-conditions require any specific information about the closed-loop system, and thus, the result of (Paunonen and Pohjolainen, 2010, Lem. 5.6) can be used here as such.

Now assume that the controller solves the robust output regulation problem. It can be shown similarly as in the proof of (Humaloja et al., 2018, Thm. IV.8) that a controller solving the robust output regulation problem implies that the first two $\mathcal{G}$-conditions are satisfied. For the third $\mathcal{G}$-condition, we first note that as we have assumed that the controller exponentially stabilizes the closed-loop system, the condition $Z = \mathcal{R}(i\omega_k - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ is automatically satisfied for all $k \in \{-q, \ldots, q\}$ by (Paunonen and Pohjolainen, 2010, Lem. 5.7). As further $\mathcal{C}_e\Sigma \in \mathcal{L}(W, Y)$ even though here the operator $\mathcal{C}_e$ is unbounded, we can utilize (Paunonen and Pohjolainen, 2010, Lem. 5.5), which implies that if a controller has the internal model structure, then the third $\mathcal{G}$-condition is satisfied. Thus, by Theorem 2, we conclude that the controller satisfies the $\mathcal{G}$-conditions.

3.2 Construction of a robust controller

In this section, we will recall a controller structure from Hämäläinen and Pohjolainen (2002), which in the preceding reference was shown to solve the robust output regulation problem for exponentially stable boundary control systems. We analyze the internal model property of the controller presented in Hämäläinen and Pohjolainen (2002) as we know that it must satisfy the $\mathcal{G}$-conditions by Theorem 3. Furthermore, a general condition for the parameter K is given to ensure exponential stability of the closed-loop system, which we have merely assumed thus far.

In order to construct a robust controller for an arbitrary exponentially stable boundary control system, we can utilize the controller introduced in Hämäläinen and Pohjolainen (2002):

$$\begin{aligned}\mathcal{G}_1 &= \operatorname{diag}(\mathcal{G}_{1,k})_{k=-q}^{q}\\ \mathcal{G}_2 &= (\mathcal{G}_{2,k})_{k=-q}^{q}\\ K &= [K_{-q}, K_{-q+1}, \ldots, K_q],\end{aligned}$$

where

$$\mathcal{G}_{1,k} = \begin{bmatrix} i\omega_k I_Y & I_Y & \ldots & 0 \\ 0 & i\omega_k I_Y & I_Y & \vdots \\ \vdots & \ddots & \ddots & I_Y \\ 0 & \ldots & 0 & i\omega_k I_Y \end{bmatrix} \in \mathcal{L}(Y^{n_k})$$

$$\mathcal{G}_{2,k} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -I_Y \end{bmatrix} \in \mathcal{L}(Y, Y^{n_k})$$

$$K_k = \left[\epsilon^{n_k} K_{k,0}, \epsilon^{n_k-1} K_{k,1}, \ldots, \epsilon K_{k,n_k-1}\right] \in \mathcal{L}(Y^{n_k}, U)$$

and $\epsilon > 0$ is the tuning parameter.

It is straightforward to see that $\mathcal{N}(\mathcal{G}_2) = \{0\}$, and for an arbitrary $k \in \{-q, \ldots, q\}$, we have

$$i\omega_k - \mathcal{G}_{1,k} = \begin{bmatrix} 0 & I_Y & \ldots & 0 \\ 0 & 0 & I_Y & \vdots \\ \vdots & \ddots & \ddots & I_Y \\ 0 & \ldots & 0 & 0 \end{bmatrix} \quad \text{and}$$

$$(i\omega_k - \mathcal{G}_{1,k})^{n_k-1} = \begin{bmatrix} 0 & \ldots & 0 & I_Y \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix}.$$

by which we obtain that $\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ and $\mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k-1} \subset \mathcal{R}(i\omega_k - \mathcal{G}_1)$. Since the choice of k was arbitrary, the controller satisfies the $\mathcal{G}$-conditions.

In order to guarantee exponential stability of the closed-loop system, the parameters K_k must be chosen in such a way that the roots of the polynomial

$$\det\left(s^{n_k} + P(i\omega_k)\sum_{l=0}^{n_k-1} K_{k,l}s^l\right)$$

lie in the open left half-plane $\mathbb{C}_-$ for all $k \in \{-q, \ldots, q\}$, where $P(\cdot)$ denotes the transfer function of the plant. If $n_k = 1$, then the corresponding condition reduces to $\sigma(P(i\omega_k)K_k) \subset \mathbb{C}_+$.

With the particular choices of the controller parameters, it follows from (Hämäläinen and Pohjolainen, 2002, Sec. III.B) that the controller stabilizes the closed-loop system exponentially. Although the operator C is assumed to be bounded in the preceding reference, the same proof is in fact valid for any admissible $C \in \mathcal{L}(\mathcal{D}(A), Y)$ as the operator C only appears in the proof as $C(s-A)^{-1}$, which is bounded for all $s \in \overline{\mathbb{C}}_+$ even for $C \in \mathcal{L}(\mathcal{D}(A), Y)$. Thus, by Theorem 3 and (Hämäläinen and Pohjolainen, 2002, Sec. III.B), a controller with the preceding parameter choices solves the robust output regulation problem.

It should be noted that if the plant and the reference and disturbance signals are real-valued, then the controller of Hämäläinen and Pohjolainen (2002) also has a real-valued realization, which is obtained from the previously presented one, e.g., by a similarity transformation. We will demonstrate the construction of a real-valued controller in the next section, where tracking of a ramp signal is considered for a one-dimensional heat equation.

4. EXAMPLE

Consider the heat equation on the domain $\zeta \in [0, 1]$ given by

$$\dot{x}(\zeta, t) = \frac{\partial^2}{\partial \zeta^2} x(\zeta, t), \quad x(\zeta, 0) = x_0(\zeta) \tag{9a}$$

$$u(t) = x(1, t) \tag{9b}$$

$$w(t) = \frac{\partial}{\partial \zeta} x(0, t) \tag{9c}$$

$$y(t) = x(0, t) \tag{9d}$$

on the state space $X = L^2(0, 1)$. By defining $\mathcal{A}x = x''$ with the maximal domain $\mathcal{D}(\mathcal{A}) := H^2(0, 1)$, $\mathcal{B}x :=$

$[x(1) \quad x'(0)]^T$, and $\mathcal{C}x := x(0)$, (9) can be equivalently written as a boundary control system

$$\dot{x}(\zeta,t) = \mathcal{A}x(\zeta,t), \quad x(\zeta,0) = x_0(\zeta) \tag{10a}$$

$$\mathcal{B}x(\cdot,t) = \begin{bmatrix}1\\0\end{bmatrix} u(t) + \begin{bmatrix}0\\1\end{bmatrix} w(t) \tag{10b}$$

$$\mathcal{C}x(\cdot,t) = y(t). \tag{10c}$$

It is easy to see that the operator $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ is the generator of an exponentially stable C_0-semigroup, and that the operator $\mathcal{B}$ has a right inverse given, e.g., by $B(\zeta) = [1 \quad \zeta - 1]$. The transfer function of the system is given by $P(s) = \cosh(\sqrt{s})^{-1}$.

Let the reference signal $y_{ref}(t)$ and the disturbance signal $w(t)$ be given by

$$y_{ref}(t) = \frac{1}{5}t, \qquad w(t) = \sin(\pi t),$$

so the signal generator S needs to have single eigenvalues at $\pm i\pi$ and an eigenvalue at zero with algebraic multiplicity two and geometric multiplicity one. Based on Section 3.2, the controller parameters can be chosen as

$$\mathcal{G}_1' = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&-i\pi&0\\0&0&0&i\pi\end{bmatrix}, \qquad \mathcal{G}_2' = \begin{bmatrix}0\\-1\\-1\\-1\end{bmatrix},$$

$$K' = \epsilon[\epsilon K_{0,0}, K_{0,1}, P(-i\pi)^{-1} P(i\pi)^{-1}]$$

where, in order to ensure the stability of the closed-loop system, we must choose $K_{0,0}$ and $K_{0,1}$ in such a way that the polynomial

$$s^2 + P(0)(K_{0,1}s + K_{0,0}) = s^2 + K_{0,1}s + K_{0,0}$$

has its roots in $\mathbb{C}_-$. Thus, we may choose, e.g., $K_{0,0} = K_{0,1} = 2$. Now, in order to construct a real-valued controller, let us introduce a matrix V given by

$$V = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&i\\0&0&1&-i\end{bmatrix}$$

and transform the chosen controller parameters into $\mathcal{G}_1 := V^{-1}\mathcal{G}_1'V$, $\mathcal{G}_2 := V^{-1}\mathcal{G}_2'$ and $K := K'V$. Thus, the real-valued controller parameters are given by

$$\mathcal{G}_1 = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&\pi\\0&0&-\pi&0\end{bmatrix}, \qquad \mathcal{G}_2 = \begin{bmatrix}0\\-1\\-1\\0\end{bmatrix},$$

$$K = \epsilon[2\epsilon, 2, 2\,\mathrm{Re}(P(i\pi)^{-1}), 2\,\mathrm{Im}(P(i\pi)^{-1})].$$

The tuning parameter ϵ is chosen as $\epsilon = 0.32$, which is determined by roughly maximizing the stability margin of the closed-loop system based on a finite-difference approximation of the plant with $N = 33$ grid points. The same approximation is used in the simulation as well. For the simulation, the initial condition for the plant is given by $x_0(\zeta) = \cos\left(\frac{\pi}{2}\zeta\right)$ and the initial state of the controller is chosen as $z_0 = 0$.

The simulation results for the heat equation are displayed in Figures 1 and 2. In Figure 1, the output $y(t)$ and the reference signal $y_{ref}(t)$ are presented along with the regulation error $e(t)$. It can be seen that after the transient errors, the output begins to follow the reference trajectory accurately. It should be noted that some of the initial error is due to the initial condition as $y(0) = 1$ as opposed to $y_{ref}(0) = 0$.

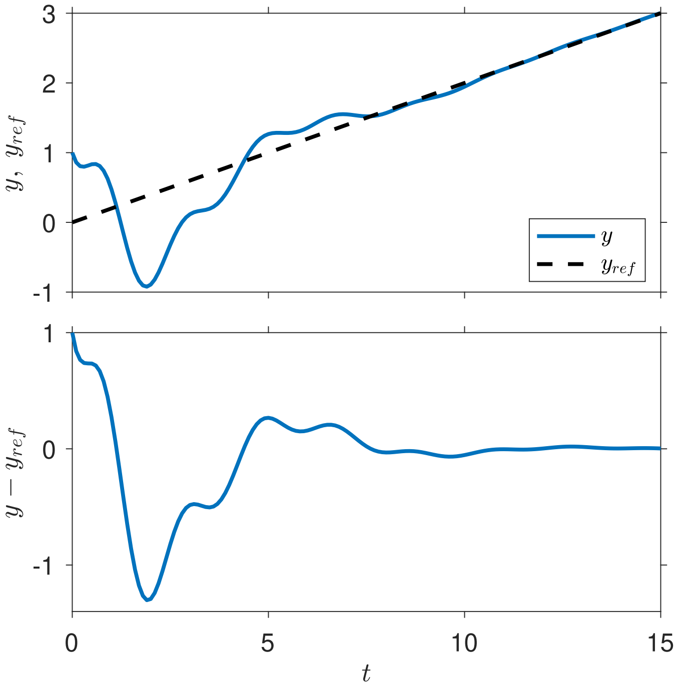

Fig. 1. Above: the reference signal y_{ref} and the output y. Below: the regulation error $e = y - y_{ref}$.

In Figure 2, the heat profile is displayed for $t \in [0, 15]$, where the output behavior can be seen along the line $\zeta = 0$. The periodic behavior along the line $\zeta = 1$ is due to the fact that the input has to account for the sinusoidal disturbance signal that acts on the heat flow at $\zeta = 0$.

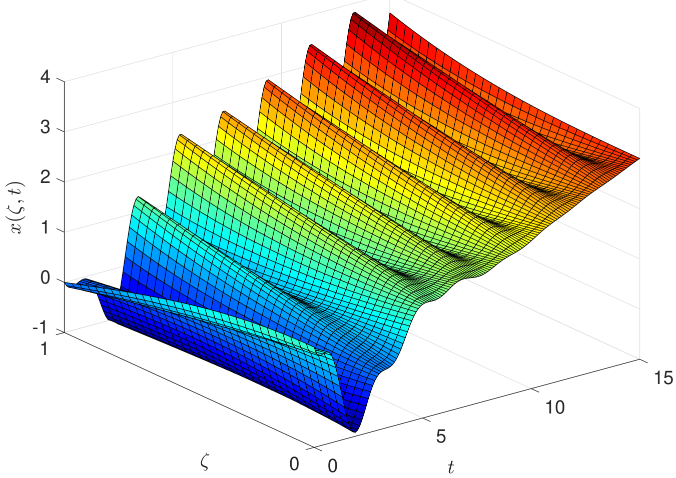

Fig. 2. The state profile of the heat equation for $t \in [0, 15]$.

5. CONCLUSIONS

We extended the internal model principle for boundary control systems to cover the case where the reference and disturbance signals contain sinusoidal parts with polynomial coefficients. The principle was presented in the form of the internal model structure and the $\mathcal{G}$-conditions, thus extending the results of Paunonen and Pohjolainen

(2010, 2014) to boundary control systems. Following the $\mathcal{G}$-conditions, the controller structure that has been presented in Hämäläinen and Pohjolainen (2002) was analyzed to solve the robust output regulation problem for the considered class of boundary control systems. As an application of the theoretic results, such a controller was constructed for the one-dimensional heat equation for robust tracking of a ramp signal.

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