On Detecting the Shape of an Unknown Object in an Electric Field

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Abstract The problem discussed in this paper is detecting the shape of an unknown object in a 2-dimensional static electric field. For simplicity, the problem is defined in a partially rectangular domain, where on a part of the boundary the potential and/or its normal derivative are known. On the other part of the boundary the boundary curve is unknown, and this curve is to be determined. The unknown part of the boundary curve describes the shape of the unknown object.
The problem is defined in the complex plane by an analytic function $w = f(z) = u(x,y) + iv(x,y)$ with the potential u as its real part. Then the inverse function is given as $f^{-1}(w) = x(u,v) + iy(u,v)$, where the functions x and y are harmonic in a rectangle with an unknown boundary condition on one boundary. The alternating-field technique is used to solve the unknown boundary condition.

1 Introduction

The problem discussed in this paper is detecting the shape of an unknown object in a 2-dimensional static electric field. For simplicity, the problem is defined in a partially rectangular domain, where on a part of the boundary the potential and its normal derivative are known, on the second part of the boundary homogeneous Neumann boundary conditions are used. On the third part of the boundary the potential is known, but the boundary curve is unknown, and this curve is to be reconstructed. The unknown part of the boundary curve describes the shape of the unknown object.

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There are quite a number of different approaches for solving such boundary reconstruction problems, e.g., the method of fundamental solutions [1], the boundary element method [4] or using an indicator function derived from Green's identities [3]. However, our efforts to apply these methods to the problem at hand have been rather unsuccessful. A more suitable method for our problem was found out to be the alternating-field technique on the inverted plane [5], where the region of the problem is conformally mapped to a rectangle in the inverted plane. In the inverted plane all the boundaries of the region are fixed, instead we have an unknown boundary condition on the boundary corresponding the free boundary in the original problem. The missing boundary condition is determined using the iterative alternating-field technique. We will adjust the technique presented in [5] to our problem and demonstrate its functionality on a few test cases.

2 Problem Formulation

Let $a,b \in \mathbb{R}$ such that $a < b$ and let $h : [a,b] \to \mathbb{R}$ such that $h \in C([a,b])$. Now, define domain R by

$$R = \{(x,y) \in \mathbb{R}^2 \,|\, x \in [a,b],\, y \in [0,h(x)]\}. \tag{1}$$

Let the lines $x = a$ and $x = b$ be perfectly insulated and the line $y = 0$ be perfectly conducting. If a constant voltage potential $u_0 = 1$ is applied to the curve $y = h(x)$, then u_0 generates the electric field $e = -\nabla u$, where the electrical potential u satisfies the following mixed boundary value problem:

$$\begin{aligned} \nabla^2 u &= 0 && \text{in } R, \\ \partial_x u &= 0 && \text{on } x = a \text{ and } x = b, \\ u &= 0 && \text{on } y = 0, \\ u &= 1 && \text{on } y = h(x). \end{aligned} \tag{2}$$

When h is known and sufficiently regular, it is well-known that the mixed boundary value problem given in Equation (2) has a unique solution. However, if h is unknown, but instead we are given an additional boundary condition $-\partial u_y = g(x)$ on the line $y = 0$ with $g : [a,b] \to \mathbb{R}$, the inverse problem of finding h is nonlinear and ill-posed. Additionally, in practice we do not actually know the entire function g but only its values at some discrete points $x_i \in [a,b]$. The geometry for the problem is displayed in Fig. 1.

3 Problem on the Inverted Plane

Since the region R is simply connected, the harmonic potential u has a harmonic conjugate v in R such that the complex potential $w = u + iv$ is analytic there. The component functions u and v are known to be connected by the Cauchy-Riemann

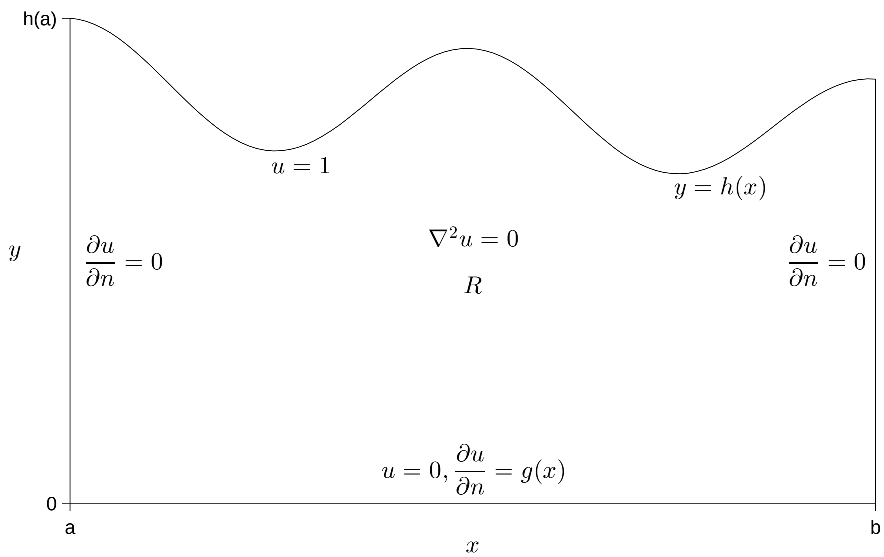

Fig. 1 *The problem on the region R.*

equations $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, and thus, it is possible to determine the boundary conditions for v from the boundary conditions of u. [2]

It can be seen directly from the Cauchy-Riemann equations that the equipotential lines of u are the lines where $\partial_n v = 0$ and conversely, the lines where $\partial_n u = 0$ are the equipotential lines of v. It yet remains to determine the values of v on the equipotential lines $x = a$ and $x = b$. From the boundary condition $\partial_n u = g(x)$ on the line $y = 0$ we obtain $g(x) = -\partial_y u = \partial_x v$ and thus, the change in the value of v between the lines $x = a$ and $x = b$ is given by $\int_a^b g(x)\mathrm{d}x$ which can be evaluated, e.g., using Simpson's rule. Since the values of v can be determined up to an additive constant, we may assign $v(x = a) = -\int_a^b g(x)\mathrm{d}x = V$ and $v(x = b) = 0$. Furthermore, we may obtain the value of v anywhere on the line $y = 0$ from $v(x) = -\int_x^b g(s)\mathrm{d}s$, which becomes necessary when we determine the boundary conditions for the inverted problem.

Now we have harmonic conjugates u and v which are real and imaginary parts of the analytic function $w = f(z) = u + iv$. If the function f is invertible in R and if $f'(z_0) \neq 0$ at each point $z_0 \in R$, then f has an analytic inverse $f^{-1}(w) = x(u,v) + iy(u,v)$ such that $f^{-1}[f(z)] = z$ [2]. Since the inverse of f is analytic in $f(R)$, its component functions x and y are harmonic conjugates there, i.e.,

$$\partial_{uu}x + \partial_{vv}x = 0 \quad \text{and} \quad \partial_{uu}y + \partial_{vv}y = 0 \tag{3}$$

and

$$\partial_u x = \partial_v y \quad \text{and} \quad \partial_v x = -\partial_u y \tag{4}$$

The region in the inverted plane is a rectangle $u \in [0,1], v \in [0,V]$ as shown in Fig. 2, where also the boundary conditions for the inverted problems are given. The boundary conditions $y = 0$, $x = a$ and $x = b$ are obtained directly from the geometry of the original problem, and the corresponding homogeneous Neumann boundary conditions are obtained from the Cauchy-Riemann equations (4). Furthermore, there is an additional boundary condition $x = x(v)$ on the line $u = 0$, which is the inverse of $v = v(x) = -\int_x^b g(s)\mathrm{d}s$. In practice the values of $x(v)$ are only required at some discrete points v_i, which can be interpolated from $v = v(x)$, e.g., by using splines.

The unknown boundary conditions $x = X(v)$ and $y = Y(v)$ on the line $u = 1$ represent the unknown boundary curve which is now mapped to a fixed line. With different values of v we will obtain points (x, y) on the z-plane, which construct the curve $y = h(x)$. The unknown boundary conditions are to be determined using the alternating-field technique which is described next.

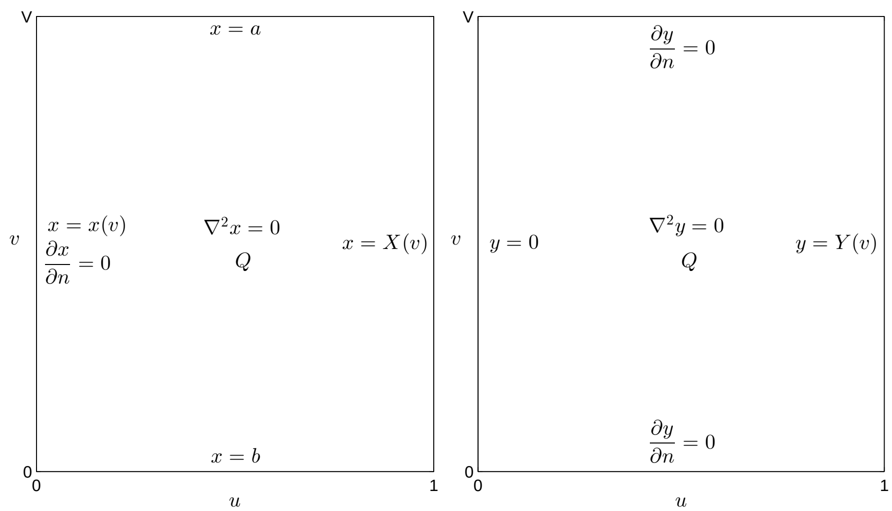

Fig. 2 The inverted problems for x and y on the region Q.

4 Alternating-Field Technique

The alternating-field technique is described in [5] by Nilson and Tsuei. The procedure given in the following is schematically similar to the one in [5], but some steps are altered due to differences in the geometries of the problems. In outline, the alternating-field technique is an iterative procedure, where we find convergent estimates for $X(v)$ and $Y(v)$ by solving Laplace's equation, by turns, for x and y. The convergence of $X(v)$ and $Y(v)$ is measured by the change in the arc length parameter s given by

$$s_i = \sum_{k=1}^{i} \sqrt{\Delta X(v_k)^2 + \Delta Y(v_k)^2}, \tag{5}$$

i.e., s is the arc length parameter of the unknown boundary curve $y = h(x)$.

For the procedure, the region Q is covered by an $M \times N$ rectangular mesh of size $\Delta u = 1/(N-1)$ and $\Delta v = V/(M-1)$. The mesh points are denoted by (u_j, v_i) such that $u_1 = 0$, $u_N = 1$, $v_1 = V$ and $v_M = 0$, and the value of x (resp. y) at a point (u_j, v_i) is denoted by x_{ij} (resp. y_{ij}). Laplace's equation is solved as a system of linear equations, where the coefficient matrix is in $\mathbb{R}^{MN \times MN}$, but it has only five nonzero diagonals. Solutions can be computed effectively by using sparse LU decomposition which needs to be computed only once for the coefficient matrices of x and y.

The steps for the iterative procedure are as follows:

0. Make an initial guess for $X(v)$. Note that $X(v_i) \in [a,b]$ for every $i \in \{1,2,\ldots,M\}$ and that $X(v_1) = a$ and $X(v_M) = b$. Then perform steps 1–6 to obtain the first iterates for $X(v)$ and $Y(v)$ and an initial estimate for the arc length parameter s.
1. Assign boundary conditions for $x(u,v)$ field, i.e., set

$$\begin{array}{ll} x_{1j} = a, \forall j \in \{1,2,\ldots,N\}, & x_{Mj} = b, \forall j \in \{1,2,\ldots,N\}, \\ x_{i1} = x(v_i), \forall i \in \{1,2,\ldots,M\} & x_{i2} = x_{i1}, \forall i \in \{1,2,\ldots,M\}, \\ x_{iN} = X(v_i), \forall i \in \{1,2,\ldots,M\}. & \end{array} \tag{6}$$

2. Solve Laplace's equation for x in Q.
3. Calculate new $Y(v)$ by the formula

$$\begin{array}{ll} Y(v_i) = -\int_0^1 \partial_v x_{ij} \mathrm{d}u, \; i \in \{2,3,\ldots,M-1\}, & \\ Y(v_1) = Y(v_2), & Y(v_M) = Y(v_{M-1}), \end{array} \tag{7}$$

   where

$$\partial_v x_{ij} \approx \frac{x_{(i-1)j} - x_{(i+1)j}}{2\Delta v} \tag{8}$$

   and the integral can be evaluated, e.g., using Simpson's rule.
4. Assign boundary conditions for $y(u,v)$ field, i.e., set

$$\begin{array}{ll} y_{1j} = y_{2j}, \forall j \in \{1,2,\ldots,N\}, & y_{Mj} = y_{(M-1)j}, \forall j \in \{1,2,\ldots,N\}, \\ y_{i1} = 0, \forall i \in \{1,2,\ldots,M\}, & y_{iN} = Y(v_i), \forall i \in \{1,2,\ldots,M\}, \end{array} \tag{9}$$

   where $Y(v_i)$ is given by Equation (7).
5. Solve Laplace's equation for y in Q.
6. Calculate new $X(v)$ by the formula

$$\begin{array}{ll} X(v_i) = x_{i1} + \int_0^1 \partial_v y_{ij} \mathrm{d}u, \; i \in \{2,3,\ldots,M-1\}, & \\ X(v_1) = a, & X(v_M) = b, \end{array} \tag{10}$$

   where

$$\partial_v y_{ij} \approx \frac{y_{(i-1)j} - y_{(i+1)j}}{2\Delta v} \tag{11}$$

and the integral can be evaluated, e.g., using Simpson's rule. Then calculate a new arc length parameter s^* from the newly obtained $X(v)$ and $Y(v)$ using Equation (5).

7. Check convergence for s, i.e., calculate $||s^* - s||^2$. If necessary, set $s = s^*$ and return to step 1. A new estimate for $X(v)$ is given by Equation (10).

The usual criterion for the procedure to stop is to see, whether $||s^* - s||^2$ is sufficiently small. Other possibility would be, e.g., to inspect the convergence rate of s and determine a suitable stopping criterion based on its changes.

5 Numerical Test Cases

The procedure described in the previous section is tested on four different boundary curves $y = h(x)$. The curves, as well as the approximations obtained from the procedure, are presented in Fig 3. In each of the cases, we have $a = 0$ and $b = 1$, and the value of $g(x)$ is computed at 21 evenly spaced points on the interval $[0, 1]$. Thus, the uv-plane is covered by a 21×21 rectangular mesh. Initial guess for $X(v)$ in all the cases is $x(v)$ and the stopping criterion for the procedure is $||s^* - s||^2 < 10^{-10}$. Data on error norms, absolute and relative maximum errors and the number of iterations required for $||s^* - s||^2 < 10^{-10}$ is displayed in Table 1 for each test case. The numbering of the cases corresponds to the order of the images in Fig 3.

Table 1 Results for the numerical test cases

Case	$\|\|Y - h(X)\|\|_2$	$\|\|Y - h(X)\|\|_\infty$	$\max \left\{ \frac{\|Y - h(X)\|}{h(X)} \right\}$	iterations
1	0.0285	0.0161	0.0424	29
2	0.0458	0.0211	0.2110	18
3	0.0320	0.0256	0.0641	21
4	0.0434	0.0379	0.0866	33

Based on Table 1 and Fig. 3 we see that for the most part the approximated boundary points Y agree with the actual boundary curve $h(X)$. However, a few noticeable errors occur as well. These errors are mostly caused by the conformal mapping from the xy-plane to the uv-plane. Namely, if $h'(0) \neq 0$, $h'(1) \neq 0$ or the curve $y = h(x)$ contains non-smooth points, those points are non-analytical points for the function $f(z) = u + iv$ and thus, the mapping $f(z)$ is not conformal at those points, which will cause errors. Most probably some errors arise from the alternating-field technique as well. However, there are no existing stability or error estimates for the technique, so these errors are virtually unknown. Regardless, it would seem that the errors caused by sources other than the conformal mapping are rather insignificant in comparison.

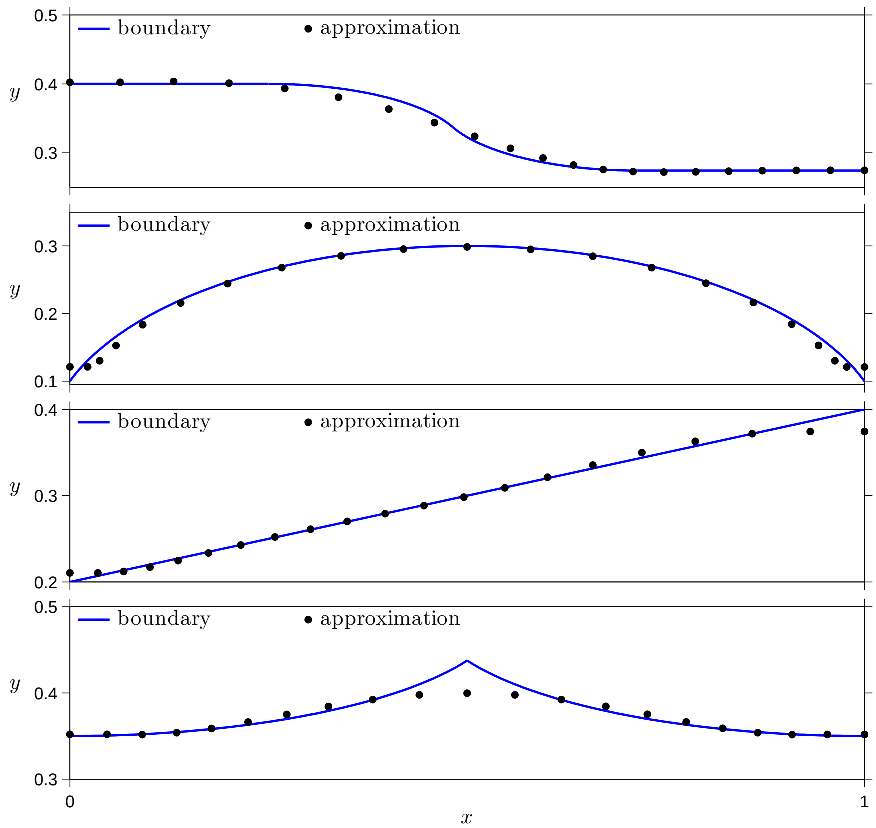

Fig. 3 The boundary curves $y = h(x)$ for the test cases $1-4$ and the approximated boundary points obtained using the alternating-field technique.

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