On the Performance of Multirate Filterbanks: Quantification of Shift Variance and Cyclostationarity in the works of Till Aach

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On the Performance of Multirate Filterbanks: Quantification of Shift Variance and Cyclostationarity in the works of Till Aach

Robert Bregovic, Atanas Gotchev
Department of Signal Processing, Tampere University of Technology, Tampere, Finland

ABSTRACT

The paper discusses the issues of shift variance and cyclostationarity in multirate filterbanks as investigated in a series of articles by Til Aach. In its first part, the paper overviews the most important properties of multirate filterbanks such as perfect reconstruction, sampling rate conversion factors, number and type of subbands and subdivisions, orthogonality and bio-orthogonality, and frequency selectivity and preservation of polynomials. This part is intended introduce the reader to the topic and make a bridge to the properties of shift variance and cyclostationarity discussed next. Criteria for shift (in)variance and cyclostationarity as derived by Til Aach are presented and commented and conclusions about their importance are made.

Keywords: Multirate filterbanks, shift-invariance, cyclostationarity, perfect reconstruction

1. INTRODUCTION

Many signal processing applications require that the signal of interest is decomposed in some transform domain where its processing is more effective or efficient. The signal reconstruction out of processed transform-domain sub-signals is then to be performed. The dual requirement, that is, the recombination of several signals into a single, compound signal, to be processed or transmitted and then split into individual components, is also valid in some applications. In digital signal processing, such decompositions and reconstructions are usually combined with sampling rate conversion. Furthermore, multirate signal processing is related with fundamental problems such as analysis of a signal in different scales and resolutions, finding multiscale signal derivatives, performing spectral analysis or transmultiplexing. Typical and important signal processing problems such as audio echo cancelation[1], multi-channel signal transmission[2,3], image noise suppression[4] rely on multirate systems for separating the information signal from the contaminating one or for combining multiple signals into one carrier signal.

Multirate signal processing systems are generally built by multirate filterbank[2,3]. In its core, a filterbank is a carefully designed set of filters that separates an input signal into several sub-signals or transform coefficients based on a given separation criteria. After processing those subband signals, a second set of filters combines the processed subband signals back into a single signal. It has been proven that in many applications by applying similar algorithms on the subband signals instead of on the original signal, considerably better results can be achieved – from the perspective of implementation complexity as well as quality of the achieved result. This made filterbanks widely spread in various areas of signal processing and established them as a methodologically very powerful tool for many, and different in first sight, applications[2,3].

While different classes of multirate filterbanks carry on methodological similarities, it is the wide variety of applications, which specify particular demands for a suitable filterbank. Correspondingly, over the last four decades a large variety of filterbanks have emerged – each having numerous, sometimes even opposing properties[5]. Based on the properties a filterbank satisfies, as well as the way a filterbank is designed and/or implemented, it is possible to identify several filterbank types, e.g. multirate filterbanks, modulated filterbanks, linear-phase filterbanks, orthogonal filterbanks. However, a "perfect recipe" that would enable a user to select the optimal filterbank for any possible application still does not exist. It is up to the user to determine which type of filterbank (having which properties) would be the best fit for his/her use scenario under consideration.

An important property related to multirate filterbanks that has not been until recently discussed much in the literature is the issue of shift-invariance[6]. In the ideal case, it would be beneficial for a filterbank to be shift invariant. This would ensure that the performance of the system (particularly from the point of subband processing) does not change when the input signal shifts. There are two main issues regarding shift invariance. First, due to the multirate nature of the filterbank, perfect shift invariance in the subbands can only be achieved for a limited set of filterbanks, as illustrated in prior work[6-12]. Second, a quantitative measure of shift-invariance is needed in order to include it in the filterbank design process as one of the constraints. In this way, at least good nearly shift-invariant filterbanks can be designed. Deriving appropriate quantitative measures has been addressed by Til Aach in his work[8-10].

The purpose of this paper is twofold. First, various filterbank properties are overviewed in an introductory fashion to help the reader with the most frequent problems in selecting a suitable filterbank. Second, an overview of Til Aach's work in the area of shift-variance for filterbanks is presented. The overview concentrates on his three journal papers covering this topic[8-10] and aims at demonstrating the significance of his work to the mentioned area.

The outline of the paper is as follows. In Section 2 an overview of most important filterbank properties is given. Section 3 discusses the shift-invariant property together with a summary of quantitate analysis of results as suggested by Aach when processing deterministic as well as random signals. Finally, some concluding remarks are presented in Section 4.

2. FILTER BANK PROPERTIES

The general block diagram of an *M*-channel filterbank is shown in Figure 1. As seen in the figure, the filterbank consists of an analysis part (analysis filterbank) and a synthesis part (synthesis filterbank). The analysis part contains set of analysis filters that decompose the signal into subbands and down-sampling blocks for reducing the sampling rate in the subbands. The synthesis part contains up-sampling blocks followed by a set of synthesis filters that enable re-assembling the subband signals into one signal.

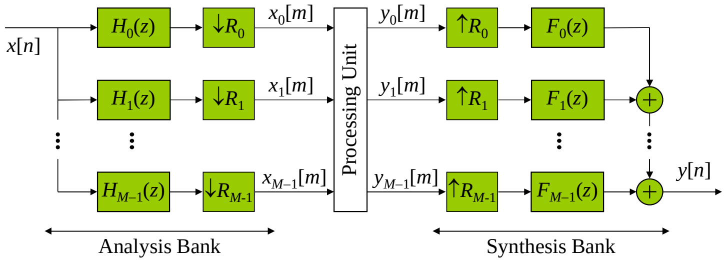

Figure 1. *M*-channel (analysis-synthesis) filterbank.

The input-output transfer function of filterbank in Figure 1 can be expressed as

$$Y(z) = T_0(z)X(z) + \sum_{l=1}^{R_k-1} T_l(z)X\left(zW_{R_k}^l\right), \tag{1}$$

where $W_R = e^{-j2\pi/R}$,

$$T_0(z) = \sum_{k=0}^{M-1} \frac{1}{R_k} H_k(z)F_k(z) \tag{2}$$

is referred to as distortion transfer function and

$$T_l(z) = \sum_{k=0}^{M-1} \frac{1}{R_k} H_k(zW_{R_k}^l)F_k(z) \tag{3}$$

is referred to as aliasing transfer function caused by the process of sampling rate conversion. Consequently, depending on the selection (design) of filters $H_k(z)$ and $F_k(z)$ and sampling rate factors R_k for $k = 0,1,\dots,M-1$, filterbanks with different properties can be obtained. Various properties are described in the following sections. It should be pointed out that the discussed properties are not exclusive – a filterbank can have various combination of properties listed in the following sections. The optimal combination of properties is very much application specific.

For the sake of simplicity, in all cases (for all properties), if not mention otherwise, it is assumed that there is no processing done on the subband signals. Furthermore, it is assumed that filters having real-valued coefficients are used. Filterbanks with filters having complex-valued coefficients are not discussed in this paper. Finally, term subband and term channel are used interchangeably.

2.1 Signal Reconstruction

One of the basic assumptions regarding the filterbank is that the filterbank itself does not alter the signal.This means that in an ideal case it is expected that the output of a filterbank is a delayed version of the input, that is,

$$y[n] = x[n-D] \tag{4}$$

with D being the filterbank delay. In the frequency domain this is equivalent to $T_0(z) = z^{-D}$ and $T_l(z) = 0$ for $l = 0,1, \dots R_k - 1$. Filterbanks satisfying (4) are known as *perfect reconstruction (PR) filterbanks*.

PR requirement is a very strict design requirement. Fortunately, in many applications it is enough if (4) is only approximately satisfied, that is,

$$y[n] \approx x[n-D]. \tag{5}$$

Such filterbanks are known as *nearly-perfect reconstruction (NPR) filterbanks*.

2.2 Amount of Data (Sampling Factors)

The down-sampling factors R_k for $k = 0,1, \dots, M-1$ determine the amount of data in the subbands. Larger R_k's result in less data in the subbands but increase the amount of aliasing (imaging) errors that are (potentially) generated in the filterbank. Smaller values of R_k's increase the amount of data but at the same time increase the redundancy in the data that is beneficial in various applications (e.g. de-noising [4]). In all cases, in order not to lose information (assuming that filters are designed correctly), the following relation must be satisfied:

$$\sum_{k=0}^{M-1} \frac{1}{R_k} \geq 1 \tag{6}$$

If (6) is not satisfied, then the input signal cannot be reconstructed from the subband signals.

A *critically-sampled multirate filterbank* is a filterbank for which the equality in (6) holds. In this case the sum of all samples in the subbands equals the number of samples in the input (output) signal – assuming a finite length input signal. All other selections of factors R_k, result in an *oversampled filterbank* – filterbank with more subband samples than needed for signal reconstruction. In the extreme case, when all R_k's are equal to one, the sampling rate in each subband is equal to the input (output) sampling rate. Such filterbanks are known as *single-rate filterbanks*.

2.3 Number of Channels

A sub-category of *M*-channel filterbanks shown in Figure 1 are the *two-channel filterbanks* ($M = 2$) shown in Figure 2. There are of special interest due to several reasons. First, they are much simpler to analyze than their *M*-channel counterpart, both, from design and implementation viewpoint. Second, they provide an easy to interpret band separation into low-pass and high-pass sections, where low-pass is related with global behavior or averaging while high-pass is related with differentiability or local details. For many applications this is quite sufficient. Finally, *M*-channel filterbanks can be constructed in a cascade manner by using two-channel filterbanks as building blocks (e.g. tree structure).

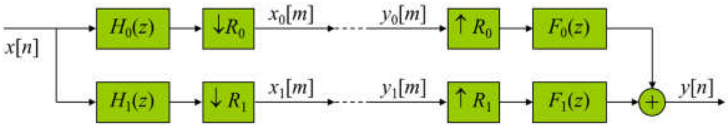

Figure 2. Two-channel filterbank.

2.4 Subband Division

Based on the selection of down-sampling factors, the frequency support of a filter bank in subbands can be uniform or non-uniform. In a *uniform filterbank*, the number of samples in each subband is equal. This is achieved by having identical down-sampling factor in all subbands, that is, $R_k = R$ for $k = 0,1,\dots,M-1$. Example of a four-channel uniform division of the frequency domain is shown in Figure 3(a).

A filterbank, in which at least two down-sampling factors are different, results in non-equal widths of subbands and is referred to as a *non-uniform filterbank*. An example of four-channel non-uniform frequency division is shown in Figure 3(b).

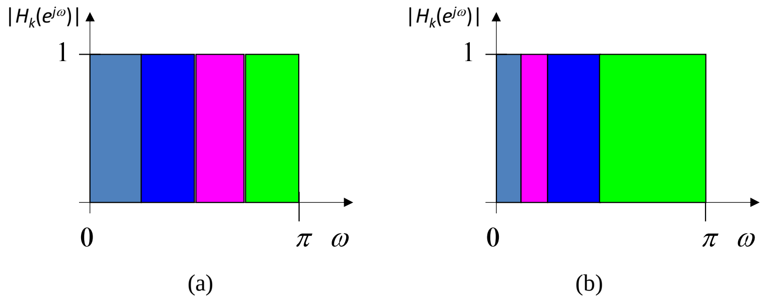

Figure 3. Some possible subband divisions for a four-channel filterbank. (a) Uniform filterbank for $R = 4$. (b) Non-uniform filterbank for $R_0 = R_1 = 8, R_2 = 4, R_3 = 2$.

2.5 Filterbank Configuration

Most commonly used filterbanks are the so-called *analysis-synthesis filterbanks* as shown in Figure 1 (when referring to a filterbank, typically an analysis-synthesis filterbank is assumed). In those filterbanks, an input signal is separated to several subband signals, which in turn are (somehow) processed, and then the processed subband signals are recombined into one signal into the synthesis stage. By exchanging the position of the analysis and synthesis bank, as shown in Figure 4, a *synthesis-analysis filterbank* is obtained (also known as *transmultiplexer*). In such filterbanks several input signals are recombined in the synthesis filterbank into one signal. This signal is, for example transmitted through some media to the analysis filterbank that reconstructs individual input signals. A typical use scenario of transmultiplexers is in communication applications – transferring several signals through a single channel.

It is worth mentioning that although filters designed for an analysis-synthesis filterbank can directly be used in a transmultiplexers, it turns out that in the some cases (particularly if NPR filterbanks are used) it is beneficially to directly design filters for usage in transmultiplexers.

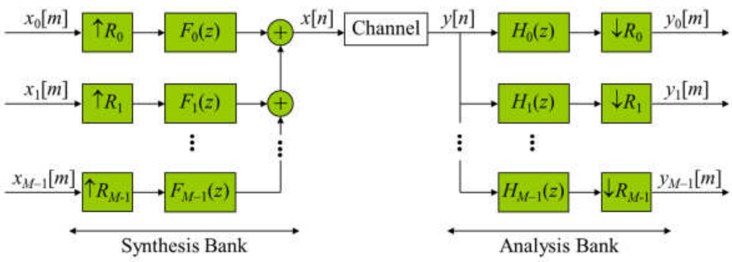

Figure 4. *M*-channel synthesis-analysis filterbank (also known as transmultiplexer).

2.6 System Delay

In a filterbank, the output signal is desired to be a delayed (PR or NPR) version of the input signal. The delay of the filterbank is directly related to the filters building the filterbank and the implementation structure. Assuming, without loss of generality, that the analysis and synthesis filters are of the same order *N*, in the most common case the filterbank delay is equal to the filter order, that is, $D = N$. Filterbanks satisfying this are known as *linear-phase filter banks*. Filterbanks in which the delay is smaller than the filter order, that is, $D < N$, are referred to as *low-delay filterbanks*.

Low-delay filterbanks are beneficial in application where the delay of the system is critical since they enable keeping the delay low and using longer filters (thereby improving other properties of the filterbank). However, low-delay filterbanks are considerably more difficult to design than their linear-phase counterparts mainly due to a higher non-linearity between the design constraints and the filter coefficients.

2.7 Frequency and Time (Space) Domain Behavior

The most common filterbanks are the so-called *frequency-selective filterbanks*. In frequency-selective filterbanks, the main goal is to design filters building the filterbank such to ensure maximum possible frequency separation between the channels. In the ideal case, this would mean no overlapping between the adjacent channels (filters). Since in practice this is not possible to achieve, the aim in designing frequency-selective filterbank will be in obtaining filters with narrow transition bandwidths and 'good' stopband attenuations. Frequency-selective filterbanks are typically used in audio and communication applications[2,3].

In some applications (e.g. image processing) more commonly used filterbanks are *filterbanks with regularities* interpreted in terms of *vanishing moments*, also known as *wavelets*. Filters building such (usually two-channel) filterbanks sacrifice frequency selectivity for smoothness in the low-pass channel. Correspondingly, a wavelet with n vanishing moments has n derivatives of the wavelet filters at zero frequency equal to zero. Vanishing moments are instrumental for representing or canceling out piecewise-polynomial components of the processed signal[13].

In comparison to aforementioned properties that are exclusive (e.g. a filterbank can be linear-phase or low-delay, but not both) it is possible to combine frequency selectivity property and regularity in a trade-off fashion, specifically when designing the synthesis (reconstruction function). Such hybrid filterbanks could inherit benefits from both categories – they are selective enough in terms of pass, transition and stop bands thereby reducing data leakage between channels and they exhibit certain regularity property that is beneficial for representing piecewise-polynomial signals of certain degree.

2.8 Filters Building the Filterbank

In addition to down-sampling and up-sampling blocks, filters are the main building blocks of a filterbank. As already commented, filters building the bank determine most properties of a filterbank. The main purpose of analysis {synthesis} filters is to reduce (eliminate as well as possible) aliasing {imaging} effects due to down-sampling {up-sampling}. Two basic classifications of filterbanks based on used filters can be made.

First, filterbanks can be built by using *Finite Impulse Response (FIR) filters* or *Infinite Impulse Response (IIR) filters*. FIR based filterbanks are in principle easier to design and are by design stable, however IIR filterbanks can achieve better properties (e.g. frequency selectivity) with a lower implementation complexity.

Second, filters building the filterbank can be *linear-phase filters* or *non-linear phase filters*. Filterbank with linear-phase filters has a constant group delay between the individual subbands and the input signal. Constant group delay preserves the shape of the signal. The main drawback of filterbanks with linear-phase filters is in the fact that the filterbank delay is directly related with the filter orders, which in turn could be problematic in applications where the filterbank delay is of importance. Filterbanks with non-linear phase filters have greater design flexibility (more degrees of freedom) but are more difficult to design.

It should be pointed out that the second property is not directly related with aforementioned linear-phase filterbank property – a linear phase filterbank can be designed by using non-linear phase filters (the cascade of non-linear phase analysis and synthesis filter can result in a linear-phase input-output response).

2.9 Orthogonality

Orthogonality implies energy preservation – sum of energy in subbands is equal to energy of the input signal. This guaranties that the energy errors in subbands (or in a transmission channel) will not be amplified by the filterbank. This is useful, for example, in communication applications. Filterbanks satisfying the orthogonality property are known as *orthogonal filterbanks*. In an orthogonal filterbank, all filters are derived from one prototype filter. Unfortunately, PR orthogonal filterbanks cannot be built by using linear-phase filters.

By relaxing the orthogonality property, a different category of filterbanks can be constructed, namely, *biorthogonal filterbanks*. Biorthogonal filterbanks are filterbanks that satisfy the orthogonality property approximately and are built by using several different linear-phase filters in the analysis filterbank (synthesis filters are derived from analysis filters).

3. SHIFT (IN)VARIANCE IN MULTIRATE SYSTEMS

A shift (time) invariant system, as shown in Figure 5, is a system for which any shifted input $x[n-m]$ results in a shifted output $y[n-m]$. A simple example of a shift-invariant system is an FIR filter with transfer function $T(z) = \sum_{k=0}^{N} h_k z^{-k}$.

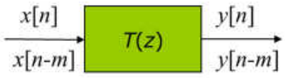

Figure 5. Shift-invariant system.

In the general case of multirate filterbanks, a shifted input $x[n-m]$ produces a shifted subband coefficients (signals) only if shift *m* is a multiplier of *M*, the number of channels. In the literature, this is referred to as periodic shift variance (or linear periodically shift variant – LPSV – since they use linear shift-invariant filters). In an ideal case, it is desired that the shift invariance holds for every *m*. This is important since multirate systems are used in various applications – for example, in compression (subband signals are grouped and quantized) or for signal and image analysis and non-linear processing (e.g. subband coefficient thresholding for de-noising and enhancement). In such cases, lack of shift invariance causes that different shifts in the input might lead to very different signals in the subbands (very different distribution of subband energy) and consequently to very different results. Furthermore, the subband processing destroys the balance between aliasing terms in subbands and as such amplifies the shift variance.

Main cause for shift variance is the presence of sampling rate changing blocks. By using over complete representations (e.g. complex wavelets[14,15], cycle spinning[16]), periodic shift invariant systems can be designed. For critically sampled filterbanks, it has been shown[7] that there is a special class of *M*-channel multirate filterbanks for which PR and perfect shift-invariance can be achieved. The authors have shown in [7] that in order for a filterbank to be shift invariant, all *polyphase* components of every filter in the filterbank must be linear-phase filters. Since this is a too restrictive requirement, it is more interesting to have quantitative criteria for estimating the amount of shift variance in a multirate filterbank, which in turn, can be used during the filterbank design together with other design criteria with the goal of, if not eliminating, then at least minimizing the amount of shift variance. Several criteria (quantitative measures for shift variance and cyclostationarity) proposed by Aach are reviewed in the following sections[8-10].

Due to the structure of a filterbank, it is logically to assume that the shift variance of the overall filterbank will be low if the shift variance of individual channel is small. Therefore, all criteria are derived for a single channel only. A block diagram of one channel of a filterbank is shown in Figure 6. Furthermore, the analysis is performed for two large classes of signals: deterministic and random wide-sense stationary (WSS) signals. The later ones are of interest when one aims at studying the expected error of an algorithm (e.g. in compression or filter approximation).

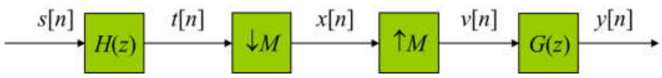

Figure 6. One channel of a multirate filterbank.

3.1 Shift Invariance for Deterministic Signals

After some multirate system expression manipulation, the energy spectra at the output of the system shown in Figure 6 (one filterbank channel), can be expressed for all possible shifts *m* as (for full step-by-step derivation see[8])

$$R_{yy}^{E}(m,z) = G(z^{-1}) R_{xx}^{E}(m,z^{m}) G(z) \tag{7}$$

with the subband energy spectra being

$$R_{xx}^{E}(m,z) = \frac{1}{M}\sum_{k=0}^{M-1} T(z^{1/M}W^{k})W^{-km} \cdot \frac{1}{M}\sum_{l=0}^{M-1} T(z^{1/M}W^{l})W^{-lm} \tag{8}$$

and

$$T(z) = H(z)S(z). \tag{9}$$

Aach proposed to estimate shift-invariance of a single channel by measuring undesired (aliasing caused) shift-variant energies $E_{yy}(m)$ at the output of the channel. This can be evaluated by integrating (8) over the unit circle:

$$E_{yy}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}^{E}(m, e^{j\omega})d\omega \tag{10}$$

By grouping $E_{yy}(m)$ into a vector and evaluating the spectra of the energies e_k one gets

$$[E_{yy}(0) \quad E_{yy}(1) \quad \dots \quad E_{yy}(M-1)]^T = W^H[e_0 \quad e_1 \quad \dots \quad e_{M-1}]^T. \tag{11}$$

The shift-invariant part of the energy is contained in the value e_0 whereas all other values are shift variant parts. Consequently, the shift-variant criteria can be expressed as

$$C_e^2 = \frac{\sum_{k=1}^{M-1}|e_k|^2}{|e_0|^2} \tag{12}$$

In a shift-invariant system, C_e^2 will be zero. A deviation from zero is a measure of shift-variance.

The criterion (12) establishes an integral measure of shift variance over the entire frequency range. As such it cannot quantify changes in spectra, which do not change the energy. Therefore, (12) was further improved in order to capture also changes in the energy spectra. The updated criteria is[8]

$$L_e^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sum_{k=1}^{M-1}\left|B_k(e^{j\omega})\right|^2}{|B_0(e^{j\omega})|^2} d\omega \tag{13}$$

with

$$\left[R_{yy}^E(0,z) \quad R_{yy}^E(1,z) \quad \dots \quad R_{yy}^E(M-1,z)\right]^T = \frac{W^H}{M^2}[B_0(z) \quad B_1(z) \quad \dots \quad B_{M-1}(z)]^T \tag{14}$$

It is seen from (12) and (13), the amount of introduced shift-variance depends mainly on the behavior of the analysis filter $H(z)$. The synthesis filter $G(z)$ attenuates the shift-variant components. Theoretically, for a very narrow synthesis filter, it would be possible to fully eliminate the shift-variance. However, this is not possible in practice since the synthesis filter is related to the analysis one through the filterbank reconstruction criteria.

Another approach for deriving a measure for shift invariance in multirate filterbanks has been presented in [10]. Given a $T \in B(\ell^2(\mathbb{Z}))_{M\mathbb{Z}}$ where $B(\ell^2(\mathbb{Z}))_{M\mathbb{Z}}$ is the subspace of $M\mathbb{Z}$-invariant operators ($B(\ell^2(\mathbb{Z}))$ is a space of bounded operators on $\ell^2(\mathbb{Z})$) and $\boldsymbol{T}$ being the associated matrix valued function uniquely characterized by the relation

$$\boldsymbol{y}_M(e^{j\omega}) = \boldsymbol{T}(e^{j\omega})\boldsymbol{s}_M(e^{j\omega}) \tag{15}$$

with $\boldsymbol{s}_M(z)$ and $\boldsymbol{y}_M(z)$ being the modulation vectors of s and y (e.g. $\boldsymbol{s}_M(z) = [S(z)\, S(zW) \dots \; S(zW^{M-1})]^T$ and $S(z)$ being the Z transform of $s[n]$). For such case, the measure for shift invariance of system T can be expressed as (for full derivation see [10]):

$$SV_2^2(T) = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} \sum_{n\neq k}\left|\boldsymbol{T}(e^{j\omega})(n,k)\right|^2 d\omega. \tag{16}$$

With other words, for a (near) shift-invariant system the matrix-valued function has to be a diagonal one, i.e. the elements that are not on the main diagonal should be zero or negligible small.

When applying the C_e^2 criteria (12) for various commonly used two-channel filterbanks, the best performance among the tested ones has been exhibited by the Haar filter based filterbank. However, the criterion L_e^2 (13) has shown that the PR conjugate quadrature filters proposed in [17] are exhibiting the best in terms of shift-invariance.

3.2 Shift Invariance for WSS Random Signals

For a WSS random signal the concept of shift does not apply. Instead, the WSS signals are characterized by their correlation functions and power spectra. A multirate system (more precisely the interpolation stage of a multirate system) turns a WSS signal to a wide-sense cyclostationarity (WSCS) signal[18,19]. Therefore, for an WSS signal the goal is to determine the cyclic nonstationarities of $y[n]$. After performing similar analysis as for deterministic signals, the criteria for evaluating the amount of cyclic nonstationarity can be expressed through the mean square power deviation from the average power as[8]

$$C_p^2 = \frac{\sum_{k=1}^{M-1}|p_k|^2}{p_0^2}, \tag{17}$$

where p_k is the power from overlapping $G(z)$ and $G(z^{-1}W^k)$ and is defined as

$$p_k = \frac{1}{2\pi M}\int_{-\pi}^{\pi} R_{xx}(e^{j\omega M})G(e^{j\omega})G(e^{-j(\omega+\frac{2\pi k}{M})})d\omega \tag{18}$$

and $R_{xx}(z)$ being the power spectrum of the down-sampled signal

$$R_{xx}(z) = \frac{1}{M}\sum_{k=0}^{M-1} H\left(z^{-1/M}W^{-k}\right)R_{ss}\left(z^{1/M}W^k\right)H\left(z^{1/M}W^k\right). \tag{19}$$

In an ideal case C_p^2 should be zero. Similar to the deterministic case, (17) evaluates the periodic variations of the power of the output, but not the variations of the shape of $R_{yy}(m,z)$. A modified criteria taking into account changes in the shape can be formulated as

$$K_p^2 = \frac{\sum_{k=1}^{M-1} D_p^2(k)}{D_p^2(0)} \tag{20}$$

with

$$D_p^2(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{R_{xx}(e^{j\omega M})}{M}G(e^{j\omega})G(e^{-j\omega}W^k)\right|^2 d\omega \tag{21}$$

In comparison to the deterministic case, the amount of nonstationarity depends on the synthesis filter $G(z)$ whereas the analysis filter $H(z)$ tends to attenuate nonstationary components. Although achieving a filterbank without nonstationary components is possible, as in the case of shift invariance, it is not practical since several other filterbank properties (discussed in Section 2) could not be met effectively.

When applying the K_p^2 criteria for various standard two-channel filterbanks assuming an AR(1)-random process as the input signal, it turns out that the filterbank with the best performance, among the tested ones (see [8] for details) is, as in the case of the criteria for deterministic signals, the PR conjugate quadrature filters of length 16 proposed in [17] with biorthogonal 5/3 filters performing worst.

3.3 Expected shift variance

A more general approach for evaluating the expected shift variance has been proposed in[10]. It can be expressed as (for full derivation see [10])

$$E_2^2(Ts) = \frac{1}{\pi}\int_{-\pi/M}^{\pi/M}\sum_{k=0}^{M-1}\sum_{l\neq k}\left|\boldsymbol{T}(e^{j\omega})(l,k)\right|^2 \times R_{ss}(e^{j\omega}W^k)\, d\omega \tag{22}$$

Where $R_{ss}(z)$ denotes the power spectrum of the WSS process *s*.

In order to illustrate the value of this estimate, it will be applied to one channel of a filterbank (c.f. Figure 6). In this case the operator *T* is given as

$$T: s \to g * (\uparrow M \downarrow M(h * s)) \tag{23}$$

and its matrix valued Fourier transform is

$$\boldsymbol{T}(z) = \frac{1}{M}\boldsymbol{g}_M(z)\boldsymbol{h}_M^T(z) \tag{24}$$

resulting into the following criterion of expected shift variance

$$E_2^2(Ts) = \frac{1}{\pi M^2} \int_{-\pi/M}^{\pi/M} \sum_{k=0}^{M-1} \sum_{l \neq k} \left| G(e^{j\omega} W^l) H(e^{j\omega} W^k) \right|^2 \times R_{ss}(e^{j\omega} W^k)\, d\omega. \quad (25)$$

In this equation, the term $G(e^{j\omega} W^l) H(e^{j\omega} W^k)$ evaluates the overlap between analysis and synthesis filter. This makes this estimate more general since, on one hand, the proposed cyclostationarity measure looks only in the synthesis filters, see e.g. (21), since they are the source of cyclostationarity, and on the other hand, the proposed shift invariant measure for deterministic signals puts emphasize on analysis filters since they are the major source of error. In contrary to those, (25) is more balanced.

Applying the above criteria for various filterbanks, it turns out that the E_2 criteria is very consistent with other criterias discussed in this and previous section. Furthermore, lower values have been achieved for longer filters, since in this case the filter provide better subband division.

4. CONCLUDING REMARKS

Multirate filterbanks are important building blocks in modern signal processing. When selecting and designing a filterbank for a given application, many properties must be taken into account. In addition to the plethora of filterbank properties that are well understood, the shift variance is an aspect of the filterbanks that requires more attention.

In this paper several criteria, proposed by Til Aach, for estimating the shift variance and cyclostationarity, have been discussed. It has been noted that the shift variance in a multirate filterbank (system) is caused by the aliasing occurring in the decimation stage whereas cyclic nonstationarities are generated by imaging occurring in the interpolation stage. The contribution of the presented works lies in the systematical approach for quantifying the shift variance and cyclostationarity. The cases of deterministic and random signals have been studied in parallel and the corresponding similarities and differences have been clarified. Furthermore, the criteria are easy to interpret and implement. Thus, they provide an additional tool for analysis of existing filterbanks and design of new ones where near shift invariance is required.

The reviewed work of Til Aach shows his deep involvement in the area of two-channel and multi-channel filterbanks and more specifically in the development of criteria for characterizing their shift variance. His work has formed a solid base for future research. Powerful tools for characterizing filter banks and selecting suitable ones for particular applications can be built using the developed criteria. As mention in [10] , the proposed shift-variance measures are quite general and more application-specific research can be performed using them as a starting point.

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