Studying the various properties of MIN and MAX matrices - elementary vs. more advanced methods

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Abstract: Let $T = \{z_1, z_2, \ldots, z_n\}$ be a finite multiset of real numbers, where $z_1 \leq z_2 \leq \cdots \leq z_n$. The purpose of this article is to study the different properties of MIN and MAX matrices of the set T with $\min(z_i, z_j)$ and $\max(z_i, z_j)$ as their ij entries, respectively. We are going to do this by interpreting these matrices as so-called meet and join matrices and by applying some known results for meet and join matrices. Once the theorems are found with the aid of advanced methods, we also consider whether it would be possible to prove these same results by using elementary matrix methods only. In many cases the answer is positive.

Keywords: MIN matrix, MAX matrix, meet matrix, join matrix

1 Introduction

MIN and MAX matrices are rather simple-structured matrices that appear in many contexts in mathematics and statistics. As is pointed out in the next section, in some cases MIN matrices have an interpretation as covariance matrices of certain stochastic processes. Bhatia [3] shows that the MIN matrix $[\min(i, j)]$ is infinitely divisible, and in [4] he gives a more comprehensive treatment to this subject. Moyé [16, Appendix B] studies the covariance matrix of Brownian motion, which appears to be a certain MIN matrix. Motivated by Moyé's work, Neudecker, Trenkler and Liu [17] defined a more general matrix

$$A = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

(a_i are real numbers for all $i = 1, \ldots, n$), and proposed the following problems:

- find a necessary and sufficient condition for A to be positive definite;
- find the inverse of A when A is nonsingular;
- find the determinant of A.

Two years later Chu, Puntanen and Styan [5] made use of elementary matrix methods and provided answers to the above questions.

Also in the field of pure mathematics MIN and MAX matrices have appeared in many contexts and by many authors. Probably the first such appearance can be found in the famous book [18] by Pólya and Szegö, where the reader is asked to calculate the determinant of the MIN matrix $[\min(i, j)]$ and also the determinants

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of some of its generalizations (in fact, all these exercises can be found already in the original German version of the book published in 1925). Meet matrices were defined by Rajarama Bhat [20] for the first time and in this same article MIN matrices are considered as an example. da Fonseca [7] studies the eigenvalues of certain MIN and MAX matrices via their matrix inverses, and in [9] bounds for the values of trigonometric functions are found by underestimating the smallest eigenvalue of a MIN matrix. Also the connection between generalized Fibonacci numbers and the characteristic polynomials of MIN and MAX matrices have been studied recently, see [2].

As we are going to see, there is a very natural and straightforward way to interpret MIN and MAX matrices as meet and join matrices, whose properties are well studied. On the other hand, because of the simple structure of MIN and MAX matrices it is easy to apply basically any result related to meet and join matrices to MIN and MAX matrices. At the same time we give some thoughts about how difficult it would be to verify these formulas by using only elementary linear algebra. The reader is also very welcome to amuse herself/himself by trying to answer the same question.

2 Preliminaries

We begin by presenting the definition of MIN and MAX matrices. Let $T = \{z_1, z_2, \ldots, z_n\}$ be a finite multiset of real numbers, where $z_1 \leq z_2 \leq \cdots \leq z_n$ (in some cases, however, we need to assume that $z_1 < z_2 < \cdots < z_n$). The MIN matrix $(T)_{\min}$ of the set T has $\min(z_i, z_j)$ as its ij entry, whereas the MAX matrix of the set T has $\max(z_i, z_j)$ as its ij entry and is denoted by $[T]_{\max}$. Both matrices are clearly square and symmetric and they may be written explicitly as

$$(T)_{\min} = \begin{bmatrix} z_1 & z_1 & z_1 & \cdots & z_1 \\ z_1 & z_2 & z_2 & \cdots & z_2 \\ z_1 & z_2 & z_3 & \cdots & z_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_n \end{bmatrix} \quad \text{and} \quad [T]_{\max} = \begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_n \\ z_2 & z_2 & z_3 & \cdots & z_n \\ z_3 & z_3 & z_3 & \cdots & z_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n & z_n & z_n & \cdots & z_n \end{bmatrix}.$$

Remark 2.1. Here it is convenient to assume that the elements of T are listed in increasing order, since this assumption does not affect most of the basic properties of the matrices $(T)_{\min}$ and $[T]_{\max}$. Rearranging the indexing of the elements of the set T corresponds to multiplying the matrices $(T)_{\min}$ and $[T]_{\max}$ from left by a certain permutation matrix Q and from right by the matrix Q^T. Properties like determinant and positive definiteness remain invariant in this operation.

An interesting special case of MIN matrices is obtained by setting $T = \{1, 2, \ldots, n\}$. In this case we have

$$(T)_{\min} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \quad \text{and} \quad [T]_{\max} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & n \\ 3 & 3 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}.$$

The matrix $(T)_{\min}$ is, up to a positive scalar, the covariance matrix of a stochastic process with increments which possess the same variance and are uncorrelated. See, for example, Davidson and MacKinnon [6, p. 606]. Bhatia [4] provided six alternative proofs for its positive definiteness. This same matrix is also studied in a recent book about matrices in statistics, see [19, pp. 251–253]. Isotalo and Puntanen [11, pp. 1021–1022] considered an example related to prediction of the new observation in the linear model with the covariance matrix of the type $(T)_{\min}$ above.

Next we review some basic concepts of lattice theory. A partially ordered set (poset) is a pair $(P, \preceq)$, where P is a nonempty set and $\preceq$ is a reflexive, antisymmetric and transitive relation. A closed interval $[x, y]$ in P is the set

$$[x, y] = \{z \in P \mid x \preceq z \preceq y\}, \quad x, y \in P.$$

Poset $(P, \preceq)$ is said to be locally finite if the interval $[x, y]$ is finite for all $x, y \in P$. Poset $(P, \preceq)$ is a chain if $x \preceq y$ or $y \preceq x$ for all $x, y \in P$. A lattice is a poset, where the infimum $x \wedge y$ and the supremum $x \vee y$ exist for all $x, y \in P$. It is easy to see that every chain is a lattice with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. For example, the set of real numbers equipped with the usual ordering is a lattice and a chain, but it is not locally finite. The set of positive integers equipped with the divisibility relation is a locally finite lattice with $x \wedge y = \gcd(x, y)$ and $x \vee y = \operatorname{lcm}(x, y)$, but this poset is not a chain. For a general account on lattices, see [21].

Next we need to define meet and join matrices. Let $(P, \preceq)$ be a locally finite lattice. Moreover, let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of P with distinct elements such that $x_i \preceq x_j \Rightarrow i \leq j$ (in other words, the indexing of the elements $x_i \in S$ is a linear extension, see [21, p. 110]). Finally, let f be a function on P to $\mathbb{R}$ (or to $\mathbb{C}$). The meet matrix $(S)_f$ of the set S with respect to the function f is the $n \times n$ matrix with $f(x_i \wedge x_j)$ as its ij entry. Similarly, the join matrix $[S]_f$ of the set S with respect to f is the $n \times n$ matrix with $f(x_i \vee x_j)$ as its ij entry. For further material about meet and join matrices we refer to [8, 12].

Like MIN and MAX matrices, meet and join matrices are square and (complex) symmetric as well. A proper way to describe meet and join matrices might be to say that in meet and join matrices the entries are determined partly by the function f and partly by the set S and the underlying lattice structure $(P, \preceq)$.

3 Some important results for meet and join matrices

In our study of MIN and MAX matrices we are going to make use of a couple of known results for meet and join matrices. The first one is about the structure of $(S)_f$. For any two subsets $S = \{x_1, x_2, \ldots, x_n\}$ and $T = \{y_1, y_2, \ldots, y_m\}$ of P, let $E(S, T) = (e_{ij})$ denote the $n \times m$ incidence matrix defined as

$$e_{ij} = \begin{cases} 1 & \text{if } y_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. [8, Theorem 1] Let $T = \{y_1, y_2, \ldots, y_m\}$ be a meet closed subset of P containing $S = \{x_1, x_2, \ldots, x_n\}$ $(m \geq n)$. Then

$$(S)_f = E \Lambda E^T = A A^T,$$

where $E = E(S, T)$, $\Lambda = \operatorname{diag}(\Psi_{T,f}(y_1), \ldots, \Psi_{T,f}(y_m))$, $A = E\Lambda^{\frac{1}{2}}$ and $\Psi_{T,f}$ is defined recursively as

$$\Psi_{T,f}(y_j) = f(y_j) - \sum_{y_i \prec y_j} \Psi_{T,f}(y_i).$$

The main idea of this factorization can be generalized for join matrices and even for meet and join matrices on two sets, see [1, Theorem 3.1] and [14, Theorem 3.1]. Furthermore, these factorization theorems can be used, among other things, to find the following determinant and inverse formulas for meet and join matrices. In Propositions 3.3 and 3.5 the function $\Phi_{S,f}$ is again the Möbius inversion of f, but in this case the inversion is executed from above. In other words,

$$\Phi_{S,f}(x_k) = f(x_k) - \sum_{x_k \prec x_v} \Phi_{S,f}(x_v).$$

Proposition 3.2. [1, Theorem 4.2] If S is meet closed, then

$$\det(S)_f = \prod_{v=1}^{n} \Psi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{\substack{z \preceq x_v \\ z \not\preceq x_t \\ t<v}} \sum_{w \preceq z} f(w)\mu_P(w, z).$$

Proposition 3.3. [14, Theorem 4.2] If S is join closed, then

$$\det[S]_f = \prod_{v=1}^{n} \Phi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{x_v \preceq x_t} f(x_t)\mu_S(x_v, x_t) = \prod_{v=1}^{n} \sum_{\substack{x_v \preceq z \preceq \vee S \\ x_t \not\preceq z \\ v<t}} \sum_{z \preceq w \preceq \vee S} f(w)\mu_P(z, w).$$

Proposition 3.4. [1, Theorem 5.3] Suppose that S is meet closed. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \sum_{k=1}^{n} \frac{(-1)^{i+j}}{\Psi_{S,f}(x_k)} \det E(S_i^k) \det E(S_j^k),$$

where $E(S_i^k)$ is the $(n-1) \times (n-1)$ submatrix of $E(S) := E(S, S)$ obtained by deleting the ith row and the kth column of $E(S)$, or

$$b_{ij} = \sum_{x_i \vee x_j \preceq x_k} \frac{\mu_S(x_i, x_k)\mu_S(x_j, x_k)}{\Psi_{S,f}(x_k)},$$

where μ_S is the Möbius function of the poset $(S, \preceq)$.

Proposition 3.5. [14, Theorem 5.3] Suppose that S is join closed. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \sum_{k=1}^{n} \frac{(-1)^{i+j}}{\Phi_{S,f}(x_k)} \det E(S_k^i) \det E(S_k^j),$$

where $E(S_k^i)$ is the $(n-1) \times (n-1)$ submatrix of $E(S)$ obtained by deleting the kth row and the ith column of $E(S)$, or

$$b_{ij} = \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu_S(x_k, x_i)\mu_S(x_k, x_j)}{\Phi_{S,f}(x_k)},$$

where μ_S is the Möbius function of the poset $(S, \preceq)$.

Our last proposition tells about the divisibility of meet and join matrices in the ring of integer matrices of size $n \times n$. Here it is also required that the values of the function f are integers.

Proposition 3.6. [13, Corollary 3.1] Let S be a chain such that $(S)_f$ is invertible (i.e. $f(x_1) \neq 0$ and $f(x_k) \neq f(x_{k-1})$ for $k = 2, 3, \ldots, n$). Then $(S)_f \mid [S]_f$.

4 MIN and MAX matrices as meet and join matrices

The most straightforward attempt to interpret MIN and MAX matrices as meet and join matrices would be to set $(P, \preceq) = (\mathbb{R}, \leq)$. This, however, cannot be done since the set of real numbers is not locally finite (meet and join matrices are usually studied via Möbius inversion, which requires the local finiteness property). Nevertheless, there is a way around the problem. We set $P = \{1, 2, \ldots, n\}$, $\preceq$ is the usual ordering $\leq$ of the integers and $S = P$. Since in this case $(P, \leq)$ is a chain with n elements, it is trivially a locally finite lattice. Moreover, by defining $f : P \to \mathbb{R}$ by $f(i) = z_i$ for all $i = 1, 2, \ldots, n$ we obtain $(S)_f = (T)_{\min}$ and $[S]_f = [T]_{\max}$.

Executing the Möbius inversion is now easy due to the simple chain-structure of the poset $(P, \leq)$ (general information about Möbius inversion and Möbius functions on posets can be found for example from [21]). For the Möbius function of the chain $(P, \leq)$ we have for $i, j \in P$ that

$$\mu_P(j, i) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The function μ_P can then be used to define two other functions Ψ_P and Φ_P as

$$\Psi_P(1) = z_1, \quad \Psi_P(i) = \sum_{1\le j\le i} \mu_P(j,i) z_j = z_i - z_{i-1} \quad \text{for } 1 < i \le n$$

and

$$\Phi_P(n) = z_n, \quad \Phi_P(i) = \sum_{i\le j\le n} \mu_P(i,j) z_j = z_i - z_{i+1} \quad \text{for } 1 \le i < n.$$

It turns out that the values of the functions Ψ_P and Φ_P characterize many key properties of the matrices $(T)_{\min}$ and $[T]_{\max}$.

Remark 4.1. Similarly defined functions $\Psi_{P,S,f}$ and $\Phi_{P,S,f}$ are also used in the study of more general meet and join matrices (see [1, Section 2], [14, Section 2] and [15, Section 2]), but here these functions take particularly simple forms due to the simple chain-structure of the set P.

Meet and join matrices and their special cases GCD and LCM matrices have been studied in dozens of research papers and their basic properties are rather well known. In this article we are going to formulate these general results for MIN and MAX matrices. Since most of the results presented in this paper follow directly from some stronger theorem for meet and join matrices, it would not be absolutely necessary to reprove these statements. However, we are going to see that in many cases it is still interesting and useful to find simpler proofs that are also accessible to those who are not so familiar with the methods used in the study of meet and join matrices.

5 Factorization of MIN and MAX matrices

Now we are in a position to give factorizations for the matrices $(T)_{\min}$ and $[T]_{\max}$ by using our newly defined functions Ψ_P and Φ_P.

Theorem 5.1. Let $n \ge 2$ and let E denote the lower triangular $0, 1$ matrix of size $n \times n$, where $e_{ij} = 1$ for $i \ge j$ and $e_{ij} = 0$ otherwise. Then

$$\begin{aligned}(T)_{\min} &= E\,\mathrm{diag}(\Psi_P(1), \Psi_P(2), \dots, \Psi_P(n))\,E^T \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 - z_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n - z_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}[T]_{\max} &= E^T\,\mathrm{diag}(\Phi_P(1), \Phi_P(2), \dots, \Phi_P(n))\,E \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 - z_2 & 0 & \cdots & 0 \\ 0 & z_2 - z_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.\end{aligned}$$

Proof. The first formula follows directly from Proposition 3.1 or from [1, Theorem 3.1], the second from [14, Theorem 3.1]. It is also easy to verify these equations by carrying out the above matrix multiplications. □

It should be noted that the factorizations found in Theorem 5.1 work also in the case when the elements of T are not distinct (i.e., $z_1 \le z_2 \le \cdots \le z_n$ and there is at least one equality). For example, if $z_{i-1} < z_i = z_{i+1} = \cdots = z_{i+k}$, then we simply have $\Psi_P(i) \neq 0$ and $\Psi_P(i+1) = \cdots = \Psi_P(i+k) = 0$.

Remark 5.1. This same factorization works also on any Hadamard power of a MIN matrix; the only thing that needs to be changed are the diagonal elements of the matrix D ($d_{11} = z_1^\alpha$, $d_{22} = z_2^\alpha - z_1^\alpha$ etc.). This factorization works even if we map all the elements z_i with an increasing real-valued function g. The explanation is that the resulting matrix remains to be a MIN matrix, but in this case the set is $\{g(z_1), g(z_2), \ldots, g(z_n)\}$.

Remark 5.2. In [5] Chu, Puntanen and Styan give a factorization, which is equivalent to that found in Theorem 5.1. Their formula can be obtained from Theorem 5.1 by multiplying the equations from left and right by the inverse matrices E^{-1} and $(E^T)^{-1}$ (the matrix $(E^T)^{-1}$ is in fact the matrix of the Möbius function μ_P of the poset $(P, \leq)$).

By taking (possibly complex-valued) square roots of the diagonal elements in Theorem 5.1 it is possible to obtain a further factorization for the matrices $(T)_{\min}$ and $[T]_{\max}$.

Theorem 5.2. Let A and B be the $n \times n$ matrices with

$$a_{ij} = \begin{cases} \sqrt{z_1} & \text{if } j = 1, \\ \sqrt{z_j - z_{j-1}} & \text{if } 1 < j \leq i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad b_{ij} = \begin{cases} \sqrt{z_i - z_{i+1}} & \text{if } j \leq i < n, \\ \sqrt{z_n} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(T)_{\min} = AA^T \quad \text{and} \quad [T]_{\max} = B^T B.$$

Proof. Let us denote

$$D = \operatorname{diag}(\Psi_P(1), \Psi_P(2), \ldots, \Psi_P(n))$$

and

$$D' = \operatorname{diag}(\Phi_P(1), \Phi_P(2), \ldots, \Phi_P(n)).$$

By Theorem 5.1 we have

$$(T)_{\min} = EDE^T = ED^{\frac{1}{2}} D^{\frac{1}{2}} E^T = (ED^{\frac{1}{2}})(ED^{\frac{1}{2}})^T = AA^T.$$

Similarly

$$[T]_{\max} = E^T D' E = E^T (D')^{\frac{1}{2}} (D')^{\frac{1}{2}} E = ((D')^{\frac{1}{2}} E)^T ((D')^{\frac{1}{2}} E) = B^T B.$$

□

Yet another factorizations for the matrices $(T)_{\min}$ and $[T]_{\max}$ can be found by making use of entrywise MIN and MAX operations for matrices. There appears to be also some resemblance between them and those in Theorem 5.2.

Theorem 5.3. Let us define the matrix operations $\wedge$ and $\vee$ by $(a_{ij})_{n\times m} \wedge (b_{ij})_{n\times m} = (\min(a_{ij}, b_{ij}))$ and $(a_{ij})_{n\times m} \vee (b_{ij})_{n\times m} = (\max(a_{ij}, b_{ij}))$. Let C denote the $n \times n$ matrix with $c_{ij} = z_i$ for all $1 \leq i, j \leq n$. Then

$$(T)_{\min} = C \wedge C^T \quad \text{and} \quad [T]_{\max} = C \vee C^T.$$

Proof. The claim follows directly after writing down the matrices $C \wedge C^T$ and $C \vee C^T$. □

6 Inertia and determinant of MIN and MAX matrices

The inertia of a Hermitean matrix A is the triple $(i_+(A), i_-(A), i_0(A))$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of positive, negative and zero eigenvalues of the matrix A, counting multiplicities. The factorization presented in Theorem 5.1 allows us to determine the inertias of the matrices $(T)_{\min}$ and $[T]_{\max}$ rather easily. We adopt the following notation: given a property P, define $\chi(P) = 1$ if P holds and $\chi(P) = 0$ otherwise.

Theorem 6.1. The inertias of the matrices $(T)_{\min}$ and $[T]_{\max}$ are given by the following formulas:
1. $i_+((T)_{\min}) = -1 + (\#\text{of distinct } z_i) + \chi(z_1 > 0)$, $i_-((T)_{\min}) = \chi(z_1 < 0)$,
2. $i_-([T]_{\max}) = -1 + (\#\text{of distinct } z_i) + \chi(z_1 < 0)$, $i_+([T]_{\max}) = \chi(z_1 > 0)$.

Proof. We prove only part (1), since part (2) follows from similar argument. The hypothesis $z_1 \leq z_2 \leq \cdots \leq z_n$ implies that exactly as many of the elements $z_2 - z_1, \ldots, z_n - z_{n-1}$ are positive as there are strict '<' signs in the chain. This number is by one less than the number of distinct elements z_i. Therefore the diagonal matrix D in Theorem 5.1, first part, has $-1 + (\#\text{of distinct } z_i)$ positive entries in rows $2, \ldots, n$. The remaining of these entries are 0. There is one additional positive, zero or negative element on the diagonal, according to if $z_1 > 0$, $z_1 = 0$ or $z_1 < 0$ holds. Thus we now have the above formulas for the inertias of the diagonal matrix D in place of $(T)_{\min}$. But Theorem 5.1 gives us that $(T)_{\min}$ is ^{T}congruent with D. Since ^{T}congruence preserves inertia by Sylvester's law, see [10, p. 223], we get the claim. □

Remark 6.1. Theorem 6.1 implies that the rank of the MIN and MAX matrices is usually the number of distinct elements in T. However, if $z_1 = 0$, then the rank of the matrix $(T)_{\min}$ decreases from this by one. The same happens to the matrix $[T]_{\max}$ if $z_n = 0$.

Next we consider the determinants of the matrices $(T)_{\min}$ and $[T]_{\max}$.

Theorem 6.2. We have

$$\det(T)_{\min} = \Psi_P(1)\Psi_P(2)\cdots\Psi_P(n) = z_1(z_2 - z_1)(z_3 - z_2)\cdots(z_n - z_{n-1})$$

and

$$\det[T]_{\max} = \Phi_P(1)\Phi_P(2)\cdots\Phi_P(n) = (z_1 - z_2)(z_2 - z_3)\cdots(z_{n-1} - z_n)z_n.$$

Proof. These determinant formulas follow directly from Proposition 3.2 and Proposition 3.3, but they are also easily recovered from Theorem 5.1. If one wishes to use elementary methods only, the Gauss-Jordan elimination process works also quite nicely but it requires a lot of computation. □

7 Inverses of MIN and MAX matrices

Under the assumption that the elements of the set T are distinct the MIN and MAX matrices of the set T are usually invertible. Next we shall find their inverses.

Theorem 7.1. Suppose that the elements of the set T are distinct. If $z_1 \neq 0$, then the MIN matrix is invertible and the inverse matrix is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \dfrac{z_2}{z_1(z_2 - z_1)} & \text{if } i = j = 1, \\ \dfrac{1}{z_i - z_{i-1}} + \dfrac{1}{z_{i+1} - z_i} & \text{if } 1 < i = j < n, \\ \dfrac{1}{z_n - z_{n-1}} & \text{if } i = j = n, \\ \dfrac{-1}{|z_i - z_j|} & \text{if } |i-j| = 1. \end{cases}$$

Similarly, if $z_n \neq 0$, then the inverse of the MAX matrix is the $n \times n$ tridiagonal matrix $C = (c_{ij})$ with

$$c_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \dfrac{1}{z_1 - z_2} & \text{if } i = j = 1, \\ \dfrac{1}{z_{i-1} - z_i} + \dfrac{1}{z_i - z_{i+1}} & \text{if } 1 < i = j < n, \\ \dfrac{1}{z_{n-1} - z_n} + \dfrac{1}{z_n} & \text{if } i = j = n, \\ \dfrac{1}{|z_i - z_j|} & \text{if } |i-j| = 1. \end{cases}$$

Proof. The inverse formulas follow straight from Proposition 3.4 and Proposition 3.5. An elementary approach would be to construct the supposed inverse matrices and multiply them with the matrices $(T)_{\min}$ and $[T]_{\max}$. □

In the case when $z_1, \ldots, z_n$ are distinct integers it is interesting to study their divisibility properties among the ring of integer matrices of size $n \times n$.

Theorem 7.2. Let $0 \neq z_1, \ldots, z_n \in \mathbb{Z}$ with $z_1 < z_2 < \cdots < z_n$. Then $(T)_{\min}$ divides $[T]_{\max}$ in the ring of $n \times n$ matrices over the integers. In other words, there exists an integer matrix A such that $[T]_{\max} = A(T)_{\min}$.

Proof. The claim follows from Proposition 3.6. Another approach would be to make use of the inverse matrix $(T)_{\min}^{-1}$ found in Theorem 7.1 and to show that the matrix $[T]_{\max}(T)_{\min}^{-1} := A$ is an integer matrix. □

8 Positive definiteness of MIN and MAX matrices

The factorizations found in Theorem 5.1 also make it possible to find conditions under which the MIN and MAX matrices of the set T are positive definite. It should be noted that Theorem 8.1 is in fact a trivial consequence of Theorem 6.1 (see Remark 8.1), but since this result can be obtained by using only elementary means, we are going to do so.

Theorem 8.1. Suppose first that the elements of the set T are distinct.
1. If $z_1 > 0$, then the matrix $(T)_{\min}$ is positive definite and the matrix $[T]_{\max}$ is indefinite.
2. If $z_1 = 0$, then the matrix $(T)_{\min}$ is positive semidefinite and the matrix $[T]_{\max}$ is indefinite.
3. If $z_1 < 0$ and $z_n > 0$, then both the matrix $(T)_{\min}$ and the matrix $[T]_{\max}$ are indefinite.
4. If $z_n = 0$, then the matrix $(T)_{\min}$ is indefinite and the matrix $[T]_{\max}$ is negative semidefinite.
5. If $z_n < 0$, then the matrix $(T)_{\min}$ is indefinite and the matrix $[T]_{\max}$ is negative definite.

If the elements of the set T are not distinct, then the matrices $(T)_{\min}$ and $[T]_{\max}$ are positive or negative semidefinite instead of positive or negative definite.

Proof. First we should note that since the 0, 1 matrix E in Theorem 5.1 is invertible, the matrices $(T)_{\min}$ and $[T]_{\max}$ share the same positive definiteness properties as the matrices

$$\mathrm{diag}(\Psi_P(1), \Psi_P(2), \ldots, \Psi_P(n))$$

and

$$\mathrm{diag}(\Phi_P(1), \Phi_P(2), \ldots, \Phi_P(n))$$

(this can be easily verified by looking at quadratic forms). Again, from the quadratic form it is easy to see that if all the diagonal elements are positive, then we have a positive definite matrix. And if all the diagonal elements are negative, then in this case the matrix is negative definite. If some of the diagonal elements are positive and some negative, then the respective matrix is indefinite. And finally, if some of the diagonal elements are equal to zero, then the matrix is positive or negative semidefinite instead of positive or negative definite (the matrix is not invertible which means that it has 0 as an eigenvalue). □

Remark 8.1. Theorem 8.1 follows easily from Theorem 6.1, since a real symmetric $n \times n$ matrix A is positive definite iff $i_+(A) = n$ and positive semidefinite iff $i_-(A) = 0$. Similarly A is negative definite iff $i_-(A) = n$ and negative semidefinite if $i_+(A) = 0$. Finally A is indefinite iff we have both $i_-(A) \geq 1$ and $i_+(A) \geq 1$.

Remark 8.2. In the case when $z_1 > 0$ and the elements of T are distinct it is also possible to prove the positive definiteness of the matrix $(T)_{\min}$ by making use of [15, Theorem 3.1]. Another option would be to use Theorem 5.2. In this case all the diagonal elements of the matrix D are positive, which implies that the matrix A is real and

$$(T)_{\min} = AA^T = AA^*.$$

The positive definiteness follows now from the invertibility of the matrix $(T)_{\min}$.

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