Asymptotic Behaviour of Platoon Systems

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***Abstract*— In this paper we study the asymptotic behaviour of various platoon-type systems using the general theory developed by the authors in a recent article. The aim is to steer an infinite number of vehicles towards a target configuration in which each vehicle has a prescribed separation from its neighbour and all vehicles are moving at a given velocity. More specifically, we study systems in which state feedback is possible, systems in which observer-based dynamic output feedback is required, and also a situation in which the control objective is modified to allow the target separations to depend on the vehicles' velocities. We show that in the first and third cases the objective can be achieved, but that in the second case the system is unstable in the sense that the associated semigroup is not uniformly bounded. We also present some quantified results concerning the rate of convergence of the platoon to its limit state when the limit exists.**

***Index Terms*— Vehicle platoon, system, ordinary differential equations, asymptotic behaviour, control, adaptive control, state feedback, rates of convergence**

I. Introduction

The purpose of this paper is to study dynamic properties the so-called *platoon system* [1–5], which describes the behaviour of an infinite chain of vehicles on a highway. The main objective is to ensure that the distances between the vehicles converge asymptotically to given target values. The behaviour of the full system is described by ordinary differential equations of the form

$$\dot{x}_k(t) = A_0 x_k(t) + A_1 x_{k-1}(t), \qquad k \in \mathbb{Z},\ t \geq 0, \tag{1}$$

where A_0 and A_1 are $m \times m$ matrices for some $m \in \mathbb{N}$ and where the initial states $x_k(0) \in \mathbb{C}^m$ for $k \in \mathbb{Z}$ are known.

The exact forms of the matrices A_0 and A_1 depend on the formulation of the control problem for the platoon system. In this paper we consider the following three different versions:

(i) In the first version we assume that state feedback can be employed in the control of the individual vehicles.

(ii) In the second version we assume that the states of the vehicles are not available for feedback, and we instead use observer-based dynamic output feedback in the control of the vehicles.

(iii) In the third version we consider a modified control objective employing a so-called *constant headway time policy* in which the target distances between the vehicles depend on the velocities of the vehicles.

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The system (1) can be written as an abstract linear differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X \tag{2}$$

on the infinite-dimensional state space $X = \ell^\infty(\mathbb{C}^m)$. The system operator A is a bounded linear operator defined by $Ax = (A_0 x_k + A_1 x_{k-1})_{k \in \mathbb{Z}}$ for all $x = (x_k)_{k \in \mathbb{Z}} \in X$. Each of the situations (i)–(iii) can be formulated in such a way that the separations between the vehicles converge to appropriate target distances if the solutions $x(t)$, $t \geq 0$, of (2) decay to zero asymptotically, i.e., $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in X$. The main purpose of this paper is to present conditions for the convergence of the solutions of (2) as $t \to \infty$. In addition we are interested in the rate of the convergence. Our results are based on application of recent theory for a more general class of infinite systems of differential equations presented in [6] and on recent developments in the theory for asymptotic behaviour of strongly continuous semigroups [7–9].

In the situation (i) we investigate the behaviour of the full platoon system under suitable stabilising state feedback control in the individual vehicles. We characterise the spectrum of A and show that semigroup generated by the system operator A is uniformly bounded. We also characterise the initial states of the full system that lead to convergent solutions and show that under additional conditions the convergence $x(t) \to z$ as $t \to \infty$ happens at a particular rational rate. In the earlier references the platoon model has been studied on the state space $X = \ell^2(\mathbb{C}^m)$, and it has in particular been shown that the system is not exponentially stabilisable [2], [5] but that strong stability can be achieved [2], [10]. Our results characterise the asymptotic behaviour of the full system on the space $X = \ell^\infty(\mathbb{C}^m)$ which can be argued to be a more realistic choice for a state space [2]. In particular, our results demonstrate that the behaviour of the platoon system on the spaces $\ell^2(\mathbb{C}^m)$ and $\ell^\infty(\mathbb{C}^m)$ differs in the respect that on the latter space some solutions do not converge at all and some of them converge to nonzero final states. This is in contrast to the fact that on $\ell^2(\mathbb{C}^m)$ the corresponding system (2) is strongly stable and all solutions satisfy $x(t) \to 0$ as $t \to \infty$.

In the situation (ii) we use identical observer-based output feedbacks to study the dynamics of the individual vehicles. We prove that regardless of the choice of the observer parameters, the system will be unstable and in particular some of the solutions $x(t)$, $t \geq 0$, of (2) will diverge at exponential rates.

Finally, in the situation (iii) the requirement for the convergence of the distances to static values is replaced by the objective which allows the target distances to depend on the velocities of the vehicles. This modification has been observed to improve the so-called *string stability* [11–13], which is often used in the study of platoon-type systems. In this paper we demonstrate that the same spacing policy also improves the stability properties of the semigroup associated to the system (2). In particular, the main stability properties of the semigroup related to the platoon system become independent of the precise locations of the eigenvalues $\sigma(A_0) \subset \mathbb{C}_-$. This is in contrast with the regular version of the platoon system, where the full system may be unstable even if $\sigma(A_0) \subset \mathbb{C}_-$, as demonstrated in Section III.

The paper is organised as follows. In Section II we recall the main results for general infinite systems of differential equations from [6]. The behaviour of the regular platoon system, that is to say version (i) of the model, is studied in Section III. The situation (ii) with dynamic output feedback is considered in Section IV, and in Section V we turn to version (iii), the platoon system with the modified spacing policy.

We use the following notation throughout the paper. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ we denote by $\ell^p(\mathbb{C}^m)$ the space of doubly infinite sequences $(x_k)_{k\in\mathbb{Z}}$ such that $x_k \in \mathbb{C}^m$ for all $k \in \mathbb{Z}$ and $\sum_{k\in\mathbb{Z}} \|x_k\|^p < \infty$ if $1 \leq p < \infty$ and $\sup_{k\in\mathbb{Z}} \|x_k\| < \infty$ if $p = \infty$. We consider $\ell^p(\mathbb{C}^m)$ with the norm given for $x = (x_k)_{k\in\mathbb{Z}}$ by $\|x\| = (\sum_{k\in\mathbb{Z}} \|x_k\|^p)^{1/p}$ if $1 \leq p < \infty$ and $\|x\| = \sup_{k\in\mathbb{Z}} \|x_k\|$ if $p = \infty$. With respect to this norm $\ell^p(\mathbb{C}^m)$ is a Banach space for $1 \leq p \leq \infty$ and a Hilbert space when $p = 2$. Moreover we write $\mathcal{B}(X)$ for the space of bounded linear operators on a Banach space X, and given $A \in \mathcal{B}(X)$ we write $\mathcal{N}(A)$ for the kernel and $\mathcal{R}(A)$ for the range of A. Moreover, we let $\sigma(A)$ and $\sigma_p(A)$ denote the spectrum and the point spectrum of A, respectively and, for $\lambda \in \mathbb{C}\setminus\sigma(A)$ we write $R(\lambda, A)$ for the resolvent operator $(\lambda - A)^{-1}$. Given two functions f and g taking values in $(0, \infty)$, we write $f(t) = O(g(t))$, $t \to \infty$, if there exists a constant $C > 0$ such that $f(t) \leq Cg(t)$ for all sufficiently large values of t. Finally, we denote the open right/left half plane by $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \gtrless 0\}$, and we use a horizontal bar over a set to denote its closure.

II. Asymptotics for General Systems

In this section we recall the main results for general infinite systems of differential equations presented in [6]. The infinite system (1) of differential equations is formulated as the abstract Cauchy problem (2) on the space $X = \ell^\infty(\mathbb{C}^m)$ with $m \in \mathbb{N}$ by choosing $x(t) = (x_k(t))_{k\in\mathbb{Z}}$ for $t \geq 0$, $x_0 = (x_k(0))_{k\in\mathbb{Z}}$ and by defining the operator $A \in \mathcal{B}(X)$ by

$$Ax = (A_0x_k + A_1x_{k-1})_{k\in\mathbb{Z}}, \qquad x = (x_k)_{k\in\mathbb{Z}} \in X.$$

Note that convergence of $x(t) \to z$ as $t \to \infty$ is equivalent to uniform convergence of all components of $x(t) = (x_k(t))_{k\in\mathbb{Z}}$ to the components of $z = (z_k)_{k\in\mathbb{Z}}$, that is

$$\sup_{k\in\mathbb{Z}} \|x_k(t) - z_k\| \to 0, \qquad t \to \infty.$$

We make the following general assumptions on the matrices A_0 and A_1. These assumptions are in particular satisfied if $\operatorname{rank} A_1 = 1$, which is true in all three formulations (i)–(iii) of the platoon system.

Assumption 2.1: We assume that

$$A_1 \neq 0. \tag{A1}$$

Moreover we assume that there exists a function ϕ such that

$$A_1R(\lambda, A_0)A_1 = \phi(\lambda)A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0). \tag{A2}$$

If this assumption is satisfied we call ϕ the *characteristic function.*

A. The spectrum of the generator

The spectrum of the operator A is determined by the characteristic function ϕ of the system (1). In particular, as will be demonstrated in Section III, the property $\sigma(A_0) \subset \mathbb{C}_-$ is in general insufficient to guarantee stability or uniform boundedness of the semigroup $T = (\exp(tA))_{t\geq0}$ generated by A.

Theorem 2.2: Suppose that (A1), (A2) hold and let

$$\Omega_\phi := \big\{\lambda \in \mathbb{C} \setminus \sigma(A_0) : |\phi(\lambda)| = 1\big\}.$$

Then the spectrum of A satisfies

$$\sigma(A) \setminus \sigma(A_0) = \Omega_\phi. \tag{3}$$

Moreover, $\sigma(A) \setminus \sigma(A_0) \subset \sigma_p(A)$ and if $\lambda \in \sigma(A) \setminus \sigma(A_0)$, then

$$\mathcal{N}(\lambda - A) = \big\{(\phi(\lambda)^kx_0)_{k\in\mathbb{Z}} : x_0 \in \mathcal{R}(R(\lambda, A_0)A_1)\big\}, \tag{4}$$

so that $\dim \mathcal{N}(\lambda - A) = \operatorname{rank}(A_1)$, and finally $\mathcal{R}(\lambda - A)$ is not dense in X.

Proof: See [6, Thm. 2.3]. ∎

Remark 2.3: As observed in [6, Rem. 2.4], the eigenvalues of A_0 may belong to either $\sigma(A)$ or $\rho(A)$.

B. The asymptotic behaviour of the semigroup

The following result presents a sufficient condition for the uniform boundedness of the semigroup T in the situation where $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$, which is characteristic for the platoon systems. For a more general sufficient condition, see [6, Thm. 3.1].

Theorem 2.4: Suppose that (A1), (A2) hold, and assume the characteristic function ϕ is of the form

$$\phi(\lambda) = \frac{\zeta^k}{(\lambda + \zeta)^k}, \qquad \lambda \in \mathbb{C} \setminus \{-\zeta\}$$

for some $\zeta > 0$ and $k \in \mathbb{N}$. Then T is uniformly bounded.

Proof: See [6, Lemma 3.2]. ∎

The results in [6] show that if the semigroup T is uniformly bounded and $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$, then the asymptotic behaviour and rates of convergence of the solutions $x(t)$, $t \geq 0$, of (2) are determined by the behaviour of the characteristic function ϕ on the imaginary axis $i\mathbb{R}$ near the origin. In particular, there exists $c > 0$ and an even integer $2 \leq n_\phi \leq 2m$ such that

$$1 - |\phi(is)| \geq c|s|^{n_\phi}, \qquad 0 < s \leq 1. \tag{5}$$

The following theorem characterises the initial states that correspond to convergent solutions $x(t)$, $t \geq 0$, and describes the rate of convergence of the derivatives $\dot{x}(t)$ as $t \to \infty$.

Theorem 2.5: Assume that (A1), (A2) hold, that $\sigma(A_0) \subset \mathbb{C}_-$ and that $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$. Suppose furthermore that the characteristic function ϕ satisfies $\phi'(0) \neq 0$ and that the semigroup T is uniformly bounded. Then the following hold:

(i) For all initial states $x_0 \in X$ the solutions $x(t)$, $t \geq 0$, of (2) satisfy

$$\|\dot{x}(t)\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right), \qquad t \to \infty$$

(ii) The solution $x(t)$, $t \geq 0$, corresponding to the initial state $x_0 = (x_k(0))_{k\in\mathbb{Z}} \in X$ converges as $t \to \infty$, i.e. there exists $z \in X$ such that $\lim_{t\to\infty} x(t) = z$, if and only if there exists $z_0 \in \mathcal{R}(A_0^{-1}A_1)$ such that

$$\sup_{j\in\mathbb{Z}} \left\| A_1 A_0^{-1}\left[z_0 - \frac{1}{n}\sum_{k=1}^{n} \phi(0)^{k-j} x_{j-k}(0)\right]\right\|_{\mathbb{C}^m} \to 0, \quad (6)$$

as $n \to \infty$. If this is true, then $z = \lim_{t\to\infty} x(t)$ is given by $z = (\phi(0)^k z_0)_{k\in\mathbb{Z}}$.

(iii) If the decay in (6) is like $O(n^{-1})$ as $n \to \infty$ then

$$\|x(t) - z\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right), \qquad t \to \infty.$$

Proof: See [6, Thm. 4.3]. ∎

III. Platoon Systems

In this section we study the regular linearised model for a platoon of vehicles. The aim is to drive the solution of the system to a configuration in which all vehicles are moving at a given constant velocity $v \in \mathbb{C}$ and the separation between the vehicles k and $k-1$ is equal to $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$ and $t \geq 0$, we write $d_k(t)$ for the separation between vehicles k and $k-1$ at time t, $v_k(t)$ for the velocity of vehicle k at time t and $a_k(t) \in \mathbb{C}$ for the acceleration of vehicle k at time t. We denote by $y_k(t) = c_k - d_k(t)$ the deviation of the actual separation from the target separation of vehicles k and $k-1$ at time t, and we let $w_k(t) = v_k(t) - v$ stand for the excess velocity of vehicle for all k at time t. Note that the variables are allowed to be complex, so that the model can be used to describe the dynamics of vehicles in the two-dimensional plane. On the other hand, if all the variables are constrained to be real, the same model can be used to study the behaviour of an infinitely long chain of vehicles.

The behaviour of the platoon system is described by the differential equations

$$\begin{pmatrix} \dot{y}_k(t) \\ \dot{w}_k(t) \\ \dot{a}_k(t) \end{pmatrix} = \begin{pmatrix} w_k(t) - w_{k-1}(t) \\ a_k(t) \\ -\tau^{-1}a_k(t) + \tau^{-1}u_k(t) \end{pmatrix}, \quad k \in \mathbb{Z}, \quad (7)$$

where $\tau > 0$ is a parameter and $u_k(t)$ is the control input of vehicle $k \in \mathbb{Z}$; see [3–5]. In this section we assume that the state variables $y_k(t)$, $w_k(t)$ and $a_k(t)$ of each of the vehicles are known for all $k \in \mathbb{Z}$ and $t \geq 0$. We stabilise the dynamics of the individual vehicles with identical state feedbacks

$$u_k(t) = -\alpha_0\tau y_k(t) - \alpha_1\tau w_k(t) + (1-\alpha_2\tau)a_k(t), \quad k \in \mathbb{Z},$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ are constants. The system (7) can then be written in the form (1) with the choices

$$x_k(t) = \begin{pmatrix} y_k(t) \\ w_k(t) \\ a_k(t) \end{pmatrix}, \quad k \in \mathbb{Z},\ t \geq 0,$$

of the states of the vehicles, and with matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\operatorname{rank} A_1 = 1$, the conditions (A1) and (A2) of Assumption 2.1 are satisfied, and the characteristic function ϕ is given by

$$\phi(\lambda) = \frac{\alpha_0}{p(\lambda)}, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0),$$

where $p(\lambda) = \lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ is the characteristic polynomial of A_0. Since $\phi(0) = 1$, it is immediate that $0 \in \sigma(A)$, and thus Theorem 2.2 implies that the platoon system cannot be stabilised exponentially. Our main goal is to choose $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ in such a way that the platoon system achieves best possible stability properties. If we restrict ourselves to real parameters α_0, α_1, and α_2, then it is well-known that the eigenvalues of A_0 belong to the half-space $\mathbb{C}_-$ if and only if $\alpha_0, \alpha_1, \alpha_2 > 0$ and $\alpha_2\alpha_1 > \alpha_0$. Figure 1 depicts the spectra of $\sigma(A)$ and $\sigma(A_0)$ for different choices of the parameters $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$. The simplest possible characteristic polynomial is $p(\lambda) = (\lambda + \zeta)^3$ with some $\zeta > 0$ corresponding to the choices $\alpha_0 = \zeta^3$, $\alpha_1 = 3\zeta^2$, and $\alpha_2 = 3\zeta$.

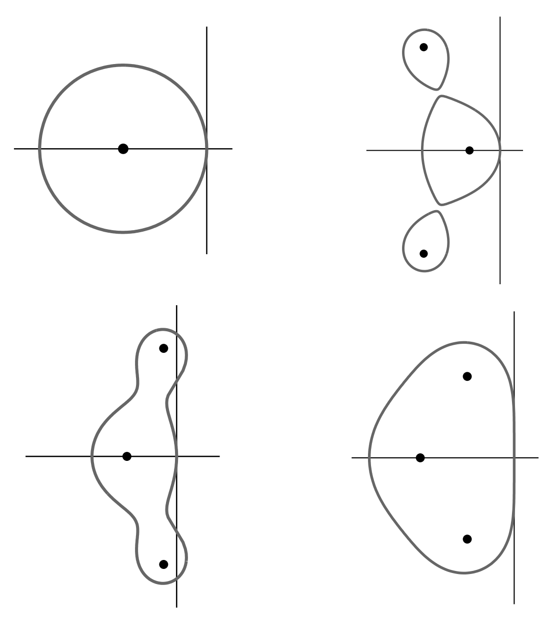

Fig. 1. The set Ω_ϕ and $\sigma(A_0)$ for four different matrices A_0.

The third graph in Figure 1 illustrates that the operator A may have unstable spectrum even if $\sigma(A_0) \subset \mathbb{C}_-$. The following proposition presents a condition for the spectrum to satisfy $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$.

Proposition 3.1: Suppose that $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ are such that $\sigma(A_0) \subset \mathbb{C}_-$. Then $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$ if and only if

$$4\alpha_1\alpha_2 > \alpha_2^3 + 8\alpha_0 \qquad \text{or} \tag{8a}$$

$$\alpha_2^4 \geq 4\alpha_1^2 \geq 2\alpha_0^3\alpha_2. \tag{8b}$$

Proof: Since $\sigma(A_0) \subset \mathbb{C}_-$, we have $\sigma(A) \cap \overline{\mathbb{C}_+} \setminus \{0\} \neq \varnothing$ if and only if there exists $s \in \mathbb{R}\setminus\{0\}$ such that $|\phi(is)| = 1$, or equivalently, $|p(is)|^2 = \alpha_0^2$. A direct computation shows that this is equivalent to

$$s^4 + (\alpha_2^2 - 2\alpha_1)s^2 + \alpha_1^2 - 2\alpha_0\alpha_2 = 0.$$

The existence of positive roots is determined by the value of the discriminant D, which satisfies

$$D = (2\alpha_1 - \alpha_2^2)^2 + 4(2\alpha_0\alpha_2 - \alpha_1^2) = \alpha_2(\alpha_2^3 - 4\alpha_1\alpha_2 + 8\alpha_0).$$

In particular, the equation has no real solutions s^2 if and only if (8a) holds, and it has only negative real solutions s^2 if and only if (8b) holds. ■

Observe that the even integer $n_\phi \geq 2$ in (5) determining the asymptotic behaviour of $x(t)$, $t \geq 0$, is the least positive integer for which there exist $c, s_0 > 0$ such that

$$|p(is)| \geq c|s|^{n_\phi} + |\alpha_0|, \qquad 0 < s \leq s_0.$$

The value of n_ϕ is uniquely determined by the locations of the eigenvalues of A_0. The following proposition shows that $n_\phi = 2$ for most spectra $\sigma(A_0)$, but also the cases $n_\phi = 4$ and $n_\phi = 6$ are possible. A case with $n_\phi = 4$ is illustrated in the last graph in Figure 1.

Proposition 3.2: Suppose that $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$. Then the following hold:

(i) $n_\phi = 4$ if and only if

$$\alpha_1^2 = 2\alpha_0\alpha_2 \quad \text{and} \quad \alpha_2^2 \neq 2\alpha_1.$$

(ii) $n_\phi = 6$ if and only if

$$\alpha_1^2 = 2\alpha_0\alpha_2 \quad \text{and} \quad \alpha_2^2 = 2\alpha_1.$$

In all other cases $n_\phi = 2$.

Proof: We have

$$|p(is)| - |\alpha_0| = \frac{|p(is)|^2 - \alpha_0^2}{|p(is)| + |\alpha_0|},$$

where $|p(is)| + |\alpha_0|$ is bounded from below and above for $s \in \mathbb{R}$ near $s = 0$. This implies that n_ϕ is the least even integer for which there exists $s_0 > 0$ and $c > 0$ such that $|p(is)|^2 - \alpha_0^2 \geq c|s|^{n_\phi}$ for $0 < s \leq s_0$. The claims of the proposition now follow from the fact that

$$|p(is)|^2 - \alpha_0^2 = s^6 + (\alpha_2^2 - 2\alpha_1)s^4 + (\alpha_1^2 - 2\alpha_0\alpha_2)s^2$$

for all $s \in \mathbb{R}$. ■

The following theorem collects our main results for the stability of the platoon system in the case where $\sigma(A_0) = \{-\zeta\}$ for some $\zeta > 0$. It should also be noted that the semigroup generated by the operator A is not contractive [6, Rem. 5.2(b)].

Theorem 3.3: If we choose $\alpha_0 = \zeta^3$, $\alpha_1 = 3\zeta^2$ and $\alpha_2 = 3\zeta$, where $\zeta > 0$ is constant, then $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$ and the semigroup T generated by A is uniformly bounded. Furthermore, the following hold:

(i) For all initial states $x_0 \in X$ the solutions $x(t)$, $t \geq 0$, of the platoon system satisfy

$$\|\dot{x}(t)\| = O\left(\left(\frac{\log t}{t}\right)^{1/2}\right), \qquad t \to \infty.$$

(ii) The solution $x(t)$ corresponding to the initial state $x_0 = (x_k(0))_{k\in\mathbb{Z}} \in X$ converges as $t \to \infty$, i.e. there exists $z \in X$ such that $\lim_{t\to\infty} x(t) = z$, if and only if there exists $c \in \mathbb{C}$ such that for the vector $(y_k(0))_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ of initial deviations we have

$$\sup_{j\in\mathbb{Z}} \left| c - \frac{1}{n}\sum_{k=1}^{n} y_{j-k}(0) \right| \to 0, \qquad n \to \infty. \tag{9}$$

If this is true, then the limit $z = \lim_{t\to\infty} x(t)$ is given by

$$z = \left(\ldots, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \ldots \right).$$

(iii) If the decay in (9) is like $O(n^{-1})$ as $n \to \infty$ then

$$\|x(t) - z\| = O\left(\left(\frac{\log t}{t}\right)^{1/2}\right), \qquad t \to \infty.$$

Proof: Note that (A1) holds and that (A2) is satisfied for the function

$$\phi(\lambda) = \frac{\zeta^3}{(\lambda + \zeta)^3}, \qquad \lambda \neq -\zeta.$$

We have $\sigma(A_0) = \{-\zeta\} \subset \mathbb{C}_-$ and

$$\Omega_\phi = \{\lambda \in \mathbb{C} : |\lambda + \zeta| = \zeta\},$$

and hence $\Omega_\phi \subset \mathbb{C}_- \cup \{0\}$. Moreover, $\phi'(0) \neq 0$ and, by Theorem 2.4, the semigroup generated by A is uniformly bounded. A simple calculation shows that $n_\phi = 2$. Noting that $\phi(0) = 1$ and that

$$A_1 A_0^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0^{-1} A_1 = \begin{pmatrix} 0 & 3/\zeta & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the claims of the theorem now follow from Theorem 2.5. ■

Remark 3.4: Our more recent results show that the logarithms in parts (i) and (iii) of Theorem 3.3 can be omitted, and thus in both cases the decay rates are $O(t^{-1/2})$.

IV. Instability of Platoon Systems with Output Feedback

In this section we consider the platoon system in a situation where the states $x_k(t)$ of the subsystems are not available for feedback, but instead the subsystems are stabilised using observer-based output feedback. We show that even though the dynamics of the individual vehicles can be stabilised with identical observers, the full system will remain unstable regardless of the choice of parameters in the observer.

If we assume that we can measure the displacements between the vehicles, the platoon system (7) can be written in the form

$$\begin{aligned}\dot{x}_k(t) &= A_0 x_k(t) + A_1 x_{k-1}(t) + B_0 u_k,\\ y_k(t) &= C_0 x_k(t),\end{aligned}$$

where $k \in \mathbb{Z}$, $t \geq 0$, and moreover

$$A_0 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & -1/\tau \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},$$

$B_0 = (0, 0, 1/\tau)^T$ and $C = (1, 0, 0)$.

The observer-based stabilising dynamic output feedback controller is given by

$$\begin{aligned}\dot{z}_k(t) &= (A_0 + LC_0) z_k(t) + B_0 u_k(t) - L y_k(t),\\ u_k(t) &= K z_k(t)\end{aligned}$$

for $k \in \mathbb{Z}$ and $t \geq 0$. Here L and K are matrices such that the eigenvalues of the matrices $A_0 + B_0 K$ and $A_0 + LC_0$ belong to $\mathbb{C}_-$. With the stabilising controller the dynamics of the vehicles can be written in the form

$$\begin{pmatrix} \dot{x}_k(t)\\ \dot{z}_k(t) \end{pmatrix} = A_0^e \begin{pmatrix} x_k(t)\\ z_k(t) \end{pmatrix} + A_1^e \begin{pmatrix} x_{k-1}(t)\\ z_{k-1}(t) \end{pmatrix}, \quad k \in \mathbb{Z}, \quad (10)$$

where

$$A_0^e = \begin{pmatrix} A_0 + LC_0 & B_0 K\\ -LC_0 & A_0 + B_0 K + LC_0 \end{pmatrix}, \; A_1^e = \begin{pmatrix} A_1 & 0\\ 0 & 0 \end{pmatrix}.$$

The spectrum of A_0^e satisfies $\sigma(A_0^e) = \sigma(A_0 + B_0 K) \cup \sigma(A_0 + LC_0)$. Since A_1^e is of rank one, there exists a characteristic function ϕ such that $A_1^e R(\lambda, A_0^e) A_1^e = \phi(\lambda) A_1^e$ for all $\lambda \in \mathbb{C} \setminus \sigma(A_0^e)$. A direct computation also shows that the characteristic funtion can be obtained from the identity

$$A_1 R_K(\lambda)(\lambda - A_0 - B_0 K - LC_0) R_L(\lambda) A_1 = \phi(\lambda) A_1,$$

where $R_K(\lambda) = R(\lambda, A_0 + B_0 K)$ and $R_L(\lambda) = R(\lambda, A_0 + LC_0)$. If $K = (k_1, k_2, k_3)$ and $L = (\ell_1, \ell_2, \ell_3)^T$ for some $k_1, k_2, k_3, \ell_1, \ell_2, \ell_3 \in \mathbb{R}$, then the characteristic function is of the form

$$\phi(\lambda) = \frac{a\lambda^2 + (b_K c_L + c_K b_L)\lambda + c_K c_L}{p_K(\lambda) p_L(\lambda)}, \quad \lambda \in \mathbb{C} \setminus \sigma(A_0^e),$$

where $a = (k_1 \ell_1 + k_2 \ell_2 + k_3 \ell_3)/\tau$ and where

$$\begin{aligned}p_K(\lambda) &= \lambda^3 + a_K \lambda^2 + b_k \lambda + c_K\\ p_L(\lambda) &= \lambda^3 + a_L \lambda^3 + b_L \lambda + c_L\end{aligned}$$

are the characteristic polynomials of the matrices $A_0 + B_0 K$ and $A_0 + LC_0$, respectively. More precisely, we have

$$\begin{aligned}a_K &= \frac{1 - k_3}{\tau}, & b_K &= -\frac{k_2}{\tau}, & c_K &= -\frac{k_1}{\tau},\\ a_L &= \frac{1}{\tau} - \ell_1, & b_L &= -\frac{\ell_1}{\tau} - \ell_2, & c_L &= -\frac{\ell_2}{\tau} - \ell_3.\end{aligned}$$

The requirements that $\sigma(A_0 + B_0 K) \subset \mathbb{C}_-$ and that $\sigma(A_0 + LC_0) \subset \mathbb{C}_-$ imply in particular that $a_K, b_K, c_K, a_L, b_L, c_L > 0$.

The following theorem shows that it is impossible to achieve stability of the full platoon system with dynamic feedback control scheme considered in this section. The state space of the full system is chosen to be $X = \ell^\infty(\mathbb{C}^6)$, but the same conclusion also holds for $X = \ell^p(\mathbb{C}^6)$ for all $1 \leq p < \infty$.

Theorem 4.1: For all choices of $k_1, k_2, k_3 \in \mathbb{R}$ and $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$ such that $\sigma(A_0 + B_0 K) \subset \mathbb{C}_-$ and $\sigma(A_0 + LC_0) \subset \mathbb{C}_-$ the infinite system of differential equations (10) is unstable in the sense that the semigroup generated by the operator

$$Ax = (A_0^e x_k + A_1^e x_{k-1})_{k \in \mathbb{Z}}, \qquad x = (x_k)_{k \in \mathbb{Z}} \in X,$$

is not uniformly bounded.

Proof: Assume that $k_1, k_2, k_3, \ell_1, \ell_2, \ell_3 \in \mathbb{R}$ are such that $\sigma(A_0 + B_0 K) \subset \mathbb{C}_-$ and $\sigma(A_0 + LC_0) \subset \mathbb{C}_-$. Then $a_K, b_K, c_K, a_L, b_L, c_L > 0$. By Theorem 2.2

$$\sigma(A) \setminus \sigma(A_0^e) = \Omega_\phi = \left\{ \lambda \in \mathbb{C} \subset \sigma(A_0^e) : |\phi(\lambda)| = 1 \right\}$$

and this part of the spectrum of A consists of eigenvalues. Our aim is to show that $\Omega_\phi \cap \mathbb{C}_+ \neq \varnothing$, which will immediately imply that the semigroup associated to the platoon system is not uniformly bounded. Clearly $\phi(0) = 1$, so $0 \in \sigma(A)$. For $\lambda \in \mathbb{C}_+$ we can write $\lambda = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi/2, \pi/2)$. We have that $|\phi(\lambda)| = 1$ precisely if $f(r, \theta) = 0$, where

$$\begin{aligned}f(r, \theta) = &\, |ar^2 e^{2i\theta} + (b_K c_L + c_K b_L) r e^{i\theta} + c_K c_L|^2\\ &- |p_K(re^{i\theta}) p_L(re^{i\theta})|^2.\end{aligned}$$

Expanding the formula for $f(r, \theta)$ shows that for a fixed θ the lowest order term in r is given by

$$-2c_K c_L (a_K c_L + a_L c_K + b_K b_L - a) \cos(2\theta) r^2.$$

If $a_K c_L + a_L c_K + b_K b_L - a \neq 0$ we can choose $\theta_0 \in (-\pi/2, \pi/2)$ so that

$$-2c_K c_L (a_K c_L + a_L c_K + b_K b_L - a) \cos(2\theta_0) > 0.$$

Then for sufficiently small $r > 0$ we necessarily have $f(r, \theta_0) > 0$. However, since the highest order term in $f(r, \theta_0)$ is equal to $-a_K^2 a_L^2 r^8$, we have $f(r, \theta_0) \to -\infty$ as $r \to \infty$. Since $r \mapsto f(r, \theta_0)$ is continuous, there exists $r_0 > 0$ such that $f(r_0, \theta_0) = 0$, and thus $|\phi(\lambda_0)| = 1$ and $\lambda_0 \in \sigma(A)$ for $\lambda_0 = r_0 e^{i\theta_0} \in \mathbb{C}_+$, which proves the claim.

It remains to consider the case where $a_K c_L + a_L c_K + b_K b_L - a = 0$. In this situation the lowest order term in r of $f(r, \theta)$ will be equal to

$$-2c_K c_L (a_K b_L + a_L b_K) \cos(3\theta) r^3$$

where $2c_K c_L (a_K b_L + a_L b_K) > 0$. If we choose $\theta_0 \in (-\pi/2, \pi/2)$ so that $\cos(3\theta_0) < 0$, we can prove the existence of $r_0 > 0$ such that $f(r_0, \theta_0) = 0$ as above, and we again have $\lambda_0 = r_0 e^{i\theta_0} \in \sigma(A) \cap \mathbb{C}_+$. ∎

V. Constant Headway Time Policy

In this final section we consider the platoon system with a modified control objective studied in [3], [4]. In particular, instead of aiming to drive the distances between the vehicles to fixed target values, we require that they approach $c_k + hv_k(t)$, where $h > 0$ and $c_k \in \mathbb{C}$ are constants and $v_k(t)$ is the velocity of vehicle $k \in \mathbb{Z}$ at time $t \geq 0$. This modification has been observed to improve string stability of the platoon system [11], [12]. We demonstrate that it also leads to stronger stability properties of the semigroup associated to the platoon system. In particular, we will show that in contrast to the results in Section III, the rational decay rate of the solutions of the system will be independent of the locations of the assigned eigenvalues of A_0, and the rates will always have the best possible exponent $1/n_\phi = 1/2$.

We begin by describing the dynamics of the platoon system similarly as in [3]. For $k \in \mathbb{Z}$ and $t \geq 0$, let

$$e_k(t) = y_k(t) - c_k - hv_k(t)$$

and $x_k(t) = (e_k(t), \dot{e}_k(t), \ddot{e}_k(t), u_k(t))^T$. Then it is shown in [3] that the behaviour of the platoon system is described by the equations

$$\dot{x}(t) = A_0 x_k(t) + A_1 x_{k-1}(t), \qquad k \in \mathbb{Z},\ t \geq 0, \tag{11}$$

with

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_1/\tau & -\beta_2/\tau & -(\beta_3+1)/\tau & 0 \\ \beta_1/h & \beta_2/h & \beta_3/h & -1/h \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/h \end{pmatrix}.$$

The values $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ are parameters of a feedback control law, and can be freely assigned to achieve stability of the individual vehicles. If we choose $\beta_1 = \alpha_0\tau$, $\beta_2 = \alpha_1\tau$ and $\beta_3 = \alpha_2\tau - 1$ where $\alpha_0, \alpha_1, \alpha_2 > 0$ and $\alpha_1\alpha_2 > \alpha_0$, then A_0 is of the form

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & 0 \\ \alpha_0\tau/h & \alpha_1\tau/h & (\alpha_2\tau - 1)/h & -1/h \end{pmatrix},$$

and $\sigma(A_0) \subset \mathbb{C}_-$. Since $\operatorname{rank} A_1 = 1$, Assumption 2.1 is satisfied and a direct computation shows that the characteristic function is given by

$$\phi(\lambda) = \frac{1}{h\lambda + 1}, \qquad \lambda \in \mathbb{C} \setminus \{-1/h\}.$$

By Theorem 2.2,

$$\sigma(A) \setminus \sigma(A_0) = \{\lambda \in \mathbb{C} : |h\lambda + 1| = 1\},$$

which a circle of radius $1/h$ centred at $-1/h$. On the other hand, because $\mathcal{R}(\lambda - A_0) + \mathcal{R}(A_1) \neq \mathbb{C}^4$ for $\lambda \in \sigma(A_0) \setminus \{-1/h\}$, we have from [6, Rem. 2.4] that in fact $\sigma(A_0) \setminus \{-1/h\} \subset \sigma(A)$. Hence the stability of the platoon system requires that $\sigma(A_0) \subset \mathbb{C}_-$ even though not all eigenvalues of A_0 affect the characteristic function ϕ. On the other hand, we will see that the precise locations of the other eigenvalues of A_0 besides $-1/h$ will have a smaller effect on the asymptotic behaviour of the platoon system than in the situation in Section III.

Since $\phi(\lambda) = \zeta/(\lambda + \zeta)$ for $\zeta = 1/h > 0$, we have from Theorem 2.4 that the semigroup generated by A is uniformly bounded. Moreover, a direct computation shows that $n_\phi = 2$. The main results of this section are presented in the following theorem. We let $X = \ell^\infty(\mathbb{C}^4)$.

Theorem 5.1: Suppose that $\alpha_0, \alpha_1, \alpha_2 > 0$ are such that $\alpha_1\alpha_2 > \alpha_0$. Then $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$ and the semigroup generated by A is uniformly bounded. Furthermore, the following hold:

(i) For all initial states $x_0 \in X$ the solutions $x(t)$, $t \geq 0$, of (2) satisfy

$$\|\dot{x}(t)\| = O\left(\left(\frac{\log t}{t}\right)^{1/2}\right), \qquad t \to \infty.$$

(ii) The solution $x(t)$, $t \geq 0$, corresponding to the initial state $x_0 = (x_k(0))_{k\in\mathbb{Z}} \in X$ with $x_k(0) = (e_k(0), \dot{e}_k(0), \ddot{e}_k(0), u_k(0))$ converges as $t \to \infty$, i.e. there exists $z \in X$ such that $\lim_{t\to\infty} x(t) = z$, if and only if there exists $c \in \mathbb{C}$ such that

$$\sup_{j\in\mathbb{Z}} \left| ch - \frac{1}{n} \sum_{k=1}^{n} \left[\dot{e}_{j-k}(0) + \tau\ddot{e}_{j-k}(0) + hu_{j-k}(0)\right] \right| \to 0$$

as $n \to \infty$. If this is true, then the limit $z = \lim_{t\to\infty} x(t)$ is given by

$$z = \left(\dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}, \dots \right).$$

(iii) If the decay in part (ii) is like $O(n^{-1})$ as $n \to \infty$ then

$$\|x(t) - z\| = O\left(\left(\frac{\log t}{t}\right)^{1/2}\right), \qquad t \to \infty.$$

Proof: Note that (A1) holds and that (A2) is satisfied for the function

$$\phi(\lambda) = \frac{1}{h\lambda + 1}, \qquad \lambda \in \mathbb{C} \setminus \{-1/h\}.$$

We have $\sigma(A_0) \subset \mathbb{C}_-$ due to the assumptions on $\alpha_0, \alpha_1, \alpha_2$. Moreover,

$$\Omega_\phi = \{\lambda \in \mathbb{C} : |h\lambda + 1| = 1\} \subset \mathbb{C}_- \cup \{0\}$$

and $\phi'(0) \neq 0$, and it follows from the form of the characteristic function ϕ and from Theorem 2.4 that the semigroup generated by A is uniformly bounded. A simple calculation shows that $n_\phi = 2$. Since $\phi(0) = 1$ and

$$A_1 A_0^{-1} = \frac{1}{h} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -\tau & -h \end{pmatrix},$$

$$A_0^{-1} A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the claims of the theorem follow from Theorem 2.5. ■

Remark 5.2: Our more recent results show that if $\alpha_0, \alpha_1, \alpha_2 > 0$ are chosen in such a way that $\sigma(A_0) = \{-1/h\}$, then the logarithms in parts (i) and (iii) of Theorem 5.1 can be omitted. The decay rates then become $O(t^{-1/2})$.

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